Introduction to Random Matrices	**Exercise Sheet No. 2**	15.10.2018

Exercise 1 - Poisson distribution

1. Consider the formula for the probability of a gap s between two adjacent random variables (drawn from N i.i.d. random variables) not depending on the position of one variable X_j (as it was introduced in the lecture):

$$p_N(s) = \int_\sigma p_N(s|\text{any } X = x) = N \int_\sigma dx p_N(s|X_j = x) p_X(x) \; . \tag{1}$$

Show that this spacing distribution is normalized, i.e.

$$\int_0^\infty ds p_N(s) = 1 \; . \tag{2}$$

2. Consider the Poisson distribution $p(s) = e^{-s}$ on $\sigma = \mathbb{R}_+$. Show that it is normalized, and has 1st moment equal, to 1, i.e.

$$\int_\sigma ds p(s) = 1 \quad \text{and} \quad \int_\sigma ds p(s) s = 1 \; . \tag{3}$$

Exercise 2 - Joint probability density function for $\beta = 1$ for 2×2 matrices
Suppose we have a matrix

$$H_s = \begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix} \tag{4}$$

with $x_1, x_2 \in N(0,1)$ and $x_3 \in N(0, 1/2)$.

Derive the jpdf for the ordered eigenvalues $\lambda_1 > \lambda_2$ for $\beta = 1$ starting from

$$\begin{aligned} \rho(\lambda_1, \lambda_2) = &\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx_3 \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}(x_1^2 + x_2^2) - x_3} \\ &\times \delta\left(\lambda_1 - \frac{1}{2}(x_1 + x_2 + S)\right) \delta\left(\lambda_2 - \frac{1}{2}(x_1 + x_2 - S)\right) \end{aligned} \tag{5}$$

where

$$S = \sqrt{(x_1 - x_2)^2 + 4x_3^2}$$

Recall that the result for $N = 2$ must be:

$$\rho(\lambda_1, \lambda_2) = \frac{1}{Z_{2,1}} e^{-\frac{1}{2}(\lambda_1^2 + \lambda_2^2)} |\lambda_1 - \lambda_2| \tag{6}$$

as was presented in the lecture. Compare your result for the normalization constant $Z_{2,1}$ with the result from the lecture for $N = 2$.
