Exercise 1 - Coulomb Potential in d dimensions

Consider a point charge q at the origin $\vec{r} = 0$ in $\mathbb{R}^d$. Recaling Gauss's law - a surface integral formula for the electric flux

$$\oint_{\partial S} \vec{E}(\vec{r}) \cdot d\vec{S} = 4\pi q \tag{1}$$

for the electic field $\vec{E}(\vec{r})$, ∂S the surface of a closed volume S, and $d\vec{S}$ an infinitesimal area perpendiculat to the surface ∂S.
Show that the potential $V(r = |\vec{r}|)$ for the rotationally invariant electric field $E(r) \cdot \vec{e}_r$ has the following dependence on the dimension d:

$$\begin{cases} d = 1 & V(r) \sim r \\ d = 2 & V(r) \sim \log r \\ d \geq 3 & V(r) \sim r^{-d+2} \ . \end{cases} \tag{2}$$

Exercise 2 - Large N analysis of the entropy integral

Show that for $N \to \infty$ the following integral

$$I_N[n(x)] = \int_{\mathbb{R}^{\mathbb{N}}} \prod_{j=1}^{N} dx_j \delta \left[n(x) - \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i) \right] \tag{3}$$

is given by

$$I_N[n(x)] \sim \exp \left[-N \int dx \ n(x) \ln n(x) \right] \tag{4}$$

using functional differentiation and saddle point analysis. Make use of the following representation of the (functional) δ-function

$$\delta \left[Nn(x) - \sum_{i=1}^{N} \delta(x - x_i) \right] = \int \mathcal{D}[\hat{n}(x)] \exp \left[i \int dx \hat{n}(x) \left[Nn(x) - \sum_{i=1}^{N} \delta(x - x_i) \right] \right] \tag{5}$$

Exercise 3 - Semi-circle integrals and large-N consistency check

1. Assume we have the following two integrals

$$\begin{aligned} \mathcal{F}_0[n(x)] &= \frac{1}{2}\int_J dx x^2 n(x) - \frac{1}{2}\int_{J^2} dx\ dx' n(x)n(x')\ln|x-x'| \\ \mathcal{F}_1[n(x)] &= \int_J dx n(x)\ln n(x)\ , \end{aligned} \tag{6}$$

   with $J = [-\sqrt{2}, \sqrt{2}]$. Compute those two integrals under the assumption that the equilibrium measure is given by

$$n(x) = \frac{1}{\pi}\sqrt{2-x^2}\ . \tag{7}$$

2. Determine the coefficients a_β and b_β in the asymptotic expansion of the partition function using the above integrals

$$\mathcal{Z}_{N,\beta} \approx \exp\left[\frac{\beta}{4}N^2\ln[N] + a_\beta N^2 + N\ln[N] + b_\beta N + o(N)\right]$$

   and compare the result to the asymptotic expansion of $\mathcal{Z}_{N,\beta=2}$ based on Barnes' G-function as given in lecture.
