Introduction to Random Matrices	**Exercise Sheet No. 4**	12.11.2018

Exercise 1 - Tricomi integrals

1. Recall the Tricomi Theorem

$$\mathrm{Pr}\int_a^b dx' \frac{f(x')}{x-x'} = g(x) \quad \Rightarrow \quad f(x) = \frac{C - \mathrm{Pr}\int_a^b \frac{dt}{\pi}\frac{\sqrt{(t-a)(b-t)}}{x-t} g(t)}{\pi\sqrt{(x-a)(b-x)}} \ . \tag{1}$$

   Use the Theorem with $g(t) = t$ imposing $\int dx n^\star(x) = 1$ to show

$$n^\star(x) = \frac{1}{\pi\sqrt{(x-a)(b-x)}}\left[1 - x^2 + \frac{1}{2}(a+b)x + \frac{1}{8}(b-a)^2\right] \ . \tag{2}$$

2. Evaluate the integral

$$\mathrm{Pr}\int_{-\sqrt{2}}^{\sqrt{2}} dz \frac{\sqrt{2-z^2}}{\pi(x-z)} = x \ , \tag{3}$$

   using the primitive of the integrand $F(y)$

$$F(y) = \frac{\sqrt{2-x^2}\ln(\varphi(x,y)) - \sqrt{2-x^2}\ln(x-y) + x\arcsin\left(\frac{y}{\sqrt{2}}\right) - \sqrt{2-y^2}}{\pi} \tag{4}$$

   where

$$\varphi(x,y) = \sqrt{2-x^2}\sqrt{2-y^2} - xy + 2 \ . \tag{5}$$

Exercise 2 - Tricomi integral for Wishart-Laguerre ensemble

Consider an ensemble with jpdf

$$\rho(x_1,\dots,x_N) \propto \exp\left[-\beta N \sum_{j=1}^N V(x_i)\right]\prod_{j<k}|x_j - x_k|^\beta \ . \tag{6}$$

Note that for $V(x) = x^2$ this goes back to the standard Gaussian case. Instead consider

$$V(x) = x - \alpha\ln x \quad x > 0 \ , \tag{7}$$

which will correspond to the Wishart-Laguerre ensemble. Try to solve

$$\mathrm{Pr}\int dx' \frac{n^\star(x')}{x-x'} = V'(x) \tag{8}$$

for the given $V(x)$.

Exercise 3 - Catalan numbers and moments of the semi-circle

Consider the moments defined by

$$\langle \mathrm{Tr}\, X^k \rangle = \int dx_1 \dots dx_N \rho(x_1, \dots, x_N) \sum_{j=1}^{N} x_j^k = N \int dx x^k \rho(x) \tag{9}$$

where $\rho(x_1, \dots, x_N)$ is the jpdf of eigenvalues and $\rho(x)$ is the one-point marginal for finite N. For the Gaussian case recall

$$\rho(x_1, \dots, x_N) = \frac{1}{Z_{N,\beta}} e^{-\frac{1}{2}\sum_{j=1}^{N} x_j^2} \prod_{j<k} |x_j - x_k|^\beta \tag{10}$$

and

$$\lim_{N\to\infty} \sqrt{\beta N} \rho(\sqrt{\beta N} x) = \rho_{\mathsf{SC}}(x) = \frac{1}{\pi}\sqrt{2 - x^2} \tag{11}$$

Show the following two equalities to hold

$$\lim_{N\to\infty} \frac{\langle \mathrm{Tr}\, X^{2n} \rangle}{\beta^n N^{n+1}} = \frac{1}{\pi} \int_{-\sqrt{2}}^{\sqrt{2}} dy y^{2n} \sqrt{2 - y^2} = \frac{C_n}{2^n} \ , \tag{12}$$

where C_n denotes the n'th Catalan number:

$$C_n = \frac{1}{n+1} \binom{2n}{n} \tag{13}$$
