Exercise 1 - Volume elements of matrix groups

1. Let $\mathbb{V}_N$ be the manifold formed by the orthogonal matrices $O \in O(N)$. Show that the volume element of this manifold is given by

$$\mathsf{Vol}(\mathbb{V}_N) = \int_{\mathbb{V}_N} dO = \frac{2^N \pi^{N^2/2}}{\Gamma_N(N/2)} \ . \tag{1}$$

where

$$\Gamma_N(a) = \pi^{N(N-1)/4} \prod_{j=1}^{N} \Gamma(a-(j-1)/2) \tag{2}$$

2. Let $\mathbb{V}_N$ now be the manifold formed by the unitary matrices $U \in U(N)$. Show that the volume element of this manifold is given by

$$\mathsf{Vol}(\mathbb{V}_N) = \int_{\mathbb{V}_N} dU = \frac{2^N \pi^{N^2}}{\widehat{\Gamma}_N(N)} \ . \tag{3}$$

where

$$\widehat{\Gamma}_N(a) = \pi^{N(N-1)/2} \prod_{j=1}^{N} \Gamma(a-(j-1)) \tag{4}$$

Exercise 2 - Vandermonde determinant

Recall the Vandermonde determinant, denoted by $\Delta_N(x)$:

$$\Delta_N(x_1,\dots,x_N) = \prod_{i<j}^{N} (x_j - x_i) \ . \tag{5}$$

Show that the name determinant is well deserved, meaning that we can write

$$\Delta_N(x_1,\dots,x_N) = \det\left[x_j^{i-1}\right]_{i,j=1,\dots,N} = \det\begin{pmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_N \\ \vdots & \vdots & \vdots \\ x_1^{N-1} & \dots & x_N^{N-1} \end{pmatrix} \ . \tag{6}$$
