Introduction to Random Matrices	**Exercise Sheet No. 6**	26.11.2018

Exercise 1 - Christoffel Darboux formula and contraction identities

1. Recall the formula for the kernel $K_N(x,y) = \sum_{j=0}^{N-1} P_j(x)P_j(y)$ of polynomials $P_j(x)$ orthonormal with respect to weight $w(x)$ on a subset of the real line $D \subset \mathbb{R}$.

   Show the Christoffel Darboux formula for $x \neq y$:

$$K_N(x,y) = C_N \frac{P_N(x)P_{N-1}(y) - P_N(y)P_{N-1}(x)}{x-y} \ . \tag{1}$$

2. Show the following contraction formulae for the kernel

$$\begin{aligned} \int_D d\lambda w(\lambda) K_N(\lambda,\lambda) &= N \ , \\ \int_D d\lambda w(\lambda) K_N(\lambda,x) &= 1 \ , \\ \int_D d\lambda w(\lambda) K_N(x,\lambda) K_N(\lambda,y) &= K_N(x,y) \ . \end{aligned} \tag{2}$$

Exercise 2 - Properties of Hermite polynomials

1. Recall the 3-step recurrence relation for orthonormal polynomials on subsets of $\mathbb{R}$:

$$\lambda P_k(\lambda) = C_{k+1}P_{k+1}(\lambda) + a_k P_k(\lambda) + C_k P_{k-1}(\lambda) \ . \tag{3}$$

   Determine the parameters a_k and C_k for the Hermite polynomials $H_k(\lambda)$ orthogonal with respect to $w(\lambda) = e^{-\lambda^2}$ on $\mathbb{R}$, with squared norms $h_p = \sqrt{\pi}2^{-p}p!$.

2. Show the differential equation

$$\frac{d}{dx} H_n(x) = 2n H_{n-1}(x) \tag{4}$$

   for the Hermite polynomials $H_n(x)$, using the recurrence relation and orthogonality condition.

3. Construct the first three Hermite polynomials H_0, H_1 and H_2 using the Gram-Schmidt orthonormalisation proceedure from lecture.

4. Write down the Christoffel Darboux kernel for the Hermite polynomials and evaluate the kernel $K_n(x,y)$ in the limit $x \to y$.
