Exercise 1 - Average of two characteristic polynomials

Recall the formula from lecture

$$\langle \det(\lambda \mathbb{1} - H) \det(\mu \mathbb{1} - H) \rangle_N = h_N K_{N+1}(\lambda, \mu)$$

which relates the kernel with the average of two characteristic polynomials.

Prove the above relation with the techniques used in the proof of the Heine formula in the lecture.

Exercise 2 - Dyson Theorem

Prove Dyson's Theorem that was given in leture in the following way:

Let $K(x, y)$ be integrable in both x and y such that the following properties hold

$$\int dz K(x,z)K(z,y) = K(x,y) \ , \quad \int dx K(x,x) = C \ . \tag{1}$$

Then it follows that

$$\int dx_N \det_{1 \leq i,j \leq N} [K(x_i, x_j)] = (C - N + 1) \det_{1 \leq i,j \leq N-1} [K(x_i, x_j)] \ . \tag{2}$$
