Introduction to Random Matrices	**Exercise Sheet No. 9**	17.12.2018

Exercise 1 - Pfaffian determinant

Consider an antisymmetric matrix $A = -A^T$ of even dimension $N = 2s$. From the definition of the Pfaffian determinant given in lecture prove that the following holds [induction is one possibility]:

$$\det[A] = (\mathrm{Pf}\,[A])^2.$$

Exercise 2 - The kth gap and eigenvalue probability

Recall the definition of the probability to find the k-th eigenvalue, out of N eigenvalues $\{x_1, \ldots, x_N\}$, at position $x_k = s$ from the lecture:

$$p_k^{(\beta)}(s) \equiv k\frac{N!}{(N-k)!k!}\left(\prod_{j=1}^{k-1}\int_{-\infty}^{s} dx_j\right)\left(\prod_{j=k+1}^{N}\int_{s}^{\infty} dx_j\right) P_N^{(\beta)}(x_1, \ldots, x_{k-1}, x_k = s, x_{k+1}, \ldots, x_N) \tag{1}$$

where $P_N^{(\beta)}(x_1, \ldots, x_N)$ is the jpdf of the eigenvalues and β is the Dyson index.
Recall the definition of the gap probability given in the lecture

$$E_k^{(\beta)}(s) \equiv \frac{N!}{(N-k)!k!}\left(\prod_{j=1}^{k}\int_{-\infty}^{s} dx_j\right)\left(\prod_{j=k+1}^{N}\int_{s}^{\infty} dx_j\right) P_N^{(\beta)}(x_1, \ldots, x_N) \tag{2}$$

where $P_N^{(\beta)}$ denotes the jpdf of N eigenvalues $\{x_1, \ldots x_N\}$, s denotes the gap parameter and β is the Dyson index.

1. Show that you can relate the Gap probability to the eigenvalue probability, i.e.

$$p_k^{(\beta)}(s) = -\sum_{l=0}^{k-1}\frac{\partial}{\partial s}E_l^{(\beta)}(s) \tag{3}$$

2. Show that for all k this probability is normalized to unity, i.e.

$$\int_{-\infty}^{\infty} ds\ p_k^{(\beta)}(s) = 1\ . \tag{4}$$
