PHYS30201 Mathematical Fundamentals of Quantum Mechanics: Particle Data Group Clebsch-Gordan coefficients

In a system with two contributions to angular momentum j_1 and j_2, Clebsch-Gordan coefficients are used to write states good of total angular momemtum J and z-component M, $|j_1, j_2; J, M\rangle$ or just $|J, M\rangle$, in terms of the basis $\{m_1, m_2\}$, $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$:

$$|j_1, j_2; J, M\rangle = \sum_{m_1 m_2} \langle j_1, m_1; j_2, m_2|J, M\rangle \Big(|j_1, m_1\rangle \otimes |j_2, m_2\rangle\Big) \qquad \text{and}$$

$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle = \sum_{JM} \langle j_1, m_1; j_2, m_2|J, M\rangle \, |j_1, j_2; J, M\rangle$$

where the numbers denoted by $\langle j_1, m_1; j_2, m_2|J, M\rangle$ are the Clebsch-Gordan coefficients; they vanish unless $j_1 + j_2 \geq J \geq |j_1 - j_2|$, and $m_1 + m_2 = M$. There is a conventional tabulation which can be found in various places including the Particle Data Group site, but the notation takes some explanation.

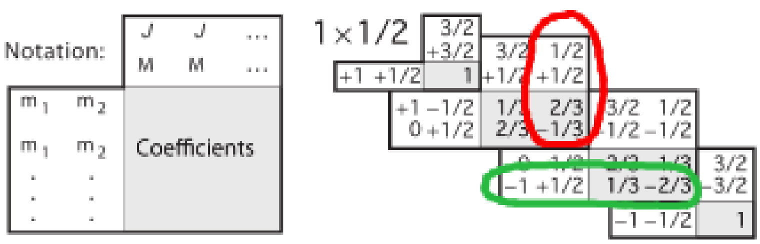
There is one table for each j_1, j_2 pair. The table consists of a series of blocks, one for each value of M. Along the top are possible values of $\begin{smallmatrix} J \\ M \end{smallmatrix}$ and at the left are possible values of $m_1 \; m_2$. Each block stands for something which could be written like this one for $j_1 = 1$, $j_2 = \frac{1}{2}$ and $M = m_1 + m_2 = \frac{1}{2}$:

m_1	m_2	J / M	$3/2$ / $+1/2$	$1/2$ / $+1/2$
$+1$	$-1/2$		$1/3$	$2/3$
0	$+1/2$		$2/3$	$-1/3$

For compactness the numbers in the blocks are the coefficients squared times their sign; thus $-\frac{1}{2}$ stands for $-\sqrt{\frac{1}{2}}$.

As an example consider the table for coupling $j_1 = 1$ and $j_2 = \frac{1}{2}$ to get $J = \frac{3}{2}$ or $\frac{1}{2}$. For clarity we will use the notation $|J, M\rangle$ in place of $|j_1, j_2; J, M\rangle$.

In red the coefficients of $|1, 1\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle$ and $|1, 0\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle$ in $|\frac{1}{2}, \frac{1}{2}\rangle$ are highlighted



$$|\tfrac{1}{2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}|1, 1\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{3}}|1, 0\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle.$$

In green are the components for the decomposition

$$|1, -1\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}|\tfrac{3}{2}, -\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}|\tfrac{1}{2}, -\tfrac{1}{2}\rangle.$$

Or for coupling $j_1 = \frac{3}{2}$ and $j_2 = 1$:

3/2×1

	5/2
	+5/2
+3/2 +1	1

	5/2	3/2
	+3/2	+3/2
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

	5/2	3/2	1/2
	+1/2	+1/2	+1/2
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

	5/2	3/2	1/2
	−1/2	−1/2	−1/2
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	−1/2

	5/2	3/2
	−3/2	−3/2
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

	5/2
	−5/2
−3/2 −1	1

giving for example

$$|\tfrac{3}{2}, -\tfrac{1}{2}\rangle = \sqrt{\tfrac{8}{15}}\,|\tfrac{3}{2}, \tfrac{1}{2}\rangle \otimes |1, -1\rangle - \sqrt{\tfrac{1}{15}}\,|\tfrac{3}{2}, -\tfrac{1}{2}\rangle \otimes |1, 0\rangle - \sqrt{\tfrac{2}{5}}\,|\tfrac{3}{2}, -\tfrac{3}{2}\rangle \otimes |1, 1\rangle$$

If instead one wants $j_1 = 1$ and $j_2 = \frac{3}{2}$, we use the relation

$$\langle j_2, m_2; j_1, m_1 | J, M\rangle = (-1)^{J-j_1-j_2}\langle j_1, m_1; j_2, m_2 | J, M\rangle$$

Note that table of Clebsch-Gordan coefficients are given for states of j_1 and j_2 coupling up to total J. But as j is a generic angular momentum, that covers s and l coupling to j, or s_1 and s_2 coupling to S etc.