Quantum Field Theory	**Exercises No. 2**	21.04.2016

[Tutor: Marc Sangel, Tutorial: Wednesday 27. April 2016, 16:00-17:30 in V2-200]

Exercise 2.1: Consider the harmonic oscillator with

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}\,, \quad \hat{a} = \sqrt{\frac{m\omega}{2}}\hat{x} + \frac{i}{\sqrt{2m\omega}}\,\hat{p}\,, \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2}}\hat{x} - \frac{i}{\sqrt{2m\omega}}\,\hat{p}\,.$$

Starting from $[\hat{x},\hat{p}] = i$ and $[\hat{x},\hat{x}] = [\hat{p},\hat{p}] = 0$, show that:

(a) $[\hat{a},\hat{a}] = [\hat{a}^\dagger,\hat{a}^\dagger] = 0\,, \quad [\hat{a},\hat{a}^\dagger] = 1\,.$

(b) $\hat{H} = \omega\Big(\hat{a}^\dagger\hat{a} + \frac{1}{2}\Big) = \frac{1}{2}\omega\Big(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\Big)\,.$

(c) $\hat{N} \equiv \hat{a}^\dagger\hat{a}$ is Hermitian (and thus has real eigenvalues).

(d) Define $\hat{N}|n\rangle \equiv n|n\rangle$ and $\langle n|n\rangle \equiv 1$. Show that this implies
$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle\,, \quad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle\,.$

Exercise 2.2: In lecture we have determined $\hat{x}_H(t)$ starting from the e.o.m. We will now determine it more directly:

(a) Use the Cauchy product formula for infinite series and induction to prove

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B}+[\hat{A},\hat{B}]+\frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]]+\frac{1}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]]+\ldots = \sum_{n=0}^{\infty}\frac{1}{n!}[\hat{A},[\ldots,[\hat{A},\hat{B}]\,.$$

(b) Determine the following operator with the help of the series expansion in a)

$$\hat{x}_H(t) \equiv e^{i\hat{H}t}\,\hat{x}\,e^{-i\hat{H}t}\,.$$

Exercise 2.3: Let us consider the Fock space of a free scalar field and define

$$\begin{aligned} |\vec{k}_1,\vec{k}_2,\ldots,\vec{k}_n\rangle &\equiv \hat{a}^\dagger_{\vec{k}_1}\hat{a}^\dagger_{\vec{k}_2}\ldots\hat{a}^\dagger_{\vec{k}_n}|0\rangle\,, \\ \hat{N} &\equiv \int \mathrm{d}^3\vec{p}\,\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}}\,, \\ :\hat{H}: &\equiv \int \mathrm{d}^3\vec{p}\,E_{\vec{p}}\,\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}}\,. \end{aligned}$$

What is then $\hat{N}|\vec{k}_1,\vec{k}_2,\ldots,\vec{k}_n\rangle$ and $:\hat{H}:|\vec{k}_1,\vec{k}_2,\ldots,\vec{k}_n\rangle$?

Exercise 2.4:

(a) Demonstrate that $|\vec{k}_1,\vec{k}_2\rangle = |\vec{k}_2,\vec{k}_1\rangle$. In other words these are Bosons.

(b) Compute $\langle\vec{k}'_1,\vec{k}'_2|\vec{k}_1,\vec{k}_2\rangle$.