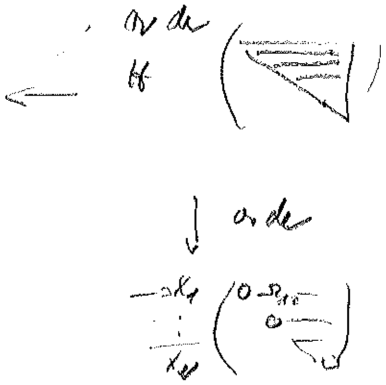
$$\boxed{\frac{dH_{11}}{dx_1}} \quad \frac{dH_{12}}{dx_1} \quad \frac{dH_{13}}{dx_1} \quad \frac{dH_{1N}}{dx_1} \quad \frac{dH_{22}}{dx_1} \quad \frac{dH_{23}}{dx_1}$$



$$\frac{dH_{11}}{d\Omega_{12}} \quad \boxed{\frac{dH_{12}}{d\Omega_{12}}}$$

$$\frac{dH_{11}}{d\Omega_{13}} \quad \frac{dH_{12}}{d\Omega_{13}} \quad \boxed{\frac{dH_{13}}{d\Omega_{13}}}$$

$$\frac{dH_{11}}{d\Omega_{1N}} \qquad \boxed{\frac{dH_{1N}}{d\Omega_{1N}}}$$

$$\frac{dH_{11}}{dx_2} \qquad \boxed{\frac{dH_{22}}{dx_2}} \quad \frac{dH_{23}}{dx_2}$$

$$\frac{dH_{11}}{d\Omega_{23}} \qquad \boxed{\frac{dH_{23}}{d\Omega_{23}}}$$

$$\frac{dH_{11}}{d\Omega_{2N}}$$

$$\frac{dH_{11}}{dx_3} \qquad \boxed{\frac{dH_{33}}{dx_3}}$$

$$\frac{dH_{11}}{d\Omega_{34}}$$

$$\vdots$$

with the result that $\left|\det\left(\frac{\partial H}{\partial x \Omega}\right)\right| = \left|\Delta_N(\{x\})\right|$

We are interested in spectral properties, e.g. distr. of indiv eigenvalue, density correlation functions.

Q: What is the probability distr. fct. of the eigenvalues?

1.3 Computation of j.p.d.f. - Eigenvalue basis

e.g. $H = O \Lambda O^T$ is a change of variables

$\Rightarrow$ Jacobian $\left[\det\left|\frac{\partial H_{ij}}{\partial(\lambda_n, O_{kl})}\right|\right]$

important: keep the same # of dof on lhs and rhs!

later

$\beta = 1$:

- $H = H^T, H_{ij} \in \mathbb{R} \Rightarrow N + \frac{N(N-1)}{2} = \frac{N(N+1)}{2}$ dof
- $\Lambda = \mathrm{diag}\, \lambda_i, \lambda_j \in \mathbb{R} \Rightarrow N$ dof
- $OO^T = 1 = O^T O$ N^2 var with cond. $\sum_{j=1}^{N} O_{ij} O_{kj} = \delta_{ik} = \delta_{ki}$ $N + \frac{N(N-1)}{2}$ cond $\Rightarrow N^2 - N - \frac{N(N-1)}{2}$ dof

$\Sigma = \frac{N(N+1)}{2}$ ✓

trick to compute Jacobian: if we can write

$$\mathrm{Tr}(dH\,dH) = \begin{pmatrix} d\lambda \\ dO \end{pmatrix} g \begin{pmatrix} d\lambda \\ dO \end{pmatrix} \quad \text{we have} \quad g = \left(\frac{\partial H_{ij}}{\partial(\lambda, O)}\right)^2$$

inv. volume el.

- g will be different for $\beta = 1, 2, 4$

→ resulting Jacobian is indep of O, U, B and only depends on β = # dof per matrix element! $|\text{Vandermonde}|^\beta$

β general:

Consider $H = S\Lambda S^\dagger \quad , SS^\dagger = S^\dagger S = \mathbb{1} \qquad \forall\ \beta = 1,2,4$

$$\Rightarrow \quad 0 = d(SS^\dagger) = dS\,S^\dagger + S\,dS^\dagger \quad , \quad S^\dagger dS + dS^\dagger S = 0$$

$$dH = dS\,\Lambda S^\dagger + S\,d\Lambda\,S^\dagger + S\Lambda\,dS^\dagger$$

$$= S\big(S^\dagger dS\,\Lambda + d\Lambda + \Lambda\,\underbrace{dS^\dagger S}_{=-S^\dagger dS}\big)S^\dagger$$

$$\boxed{dH = S\big(d\Lambda + [S^\dagger dS, \Lambda]\big)S^\dagger} \quad , [A,B] = AB - BA$$

commutator

$H = H^\dagger$

$$\Rightarrow \quad \mathrm{Tr}\left[dH\,dH^\dagger\right] = \mathrm{Tr}\left[\left(d\Lambda + [S^\dagger dS, \Lambda]\right)^2\right]$$

$\mathrm{Tr}[AB] = \mathrm{Tr}[BA]$

$$= \mathrm{Tr}\Big\{d\Lambda^2 + \underbrace{2d\Lambda[S^\dagger dS, \Lambda]}_{= 2d\Lambda\,S^\dagger dS\,\Lambda - 2d\Lambda\,\Lambda\,S^\dagger dS} + [S^\dagger dS, \Lambda]^2\Big]$$

$\xRightarrow{\mathrm{Tr\ cyclic}}$ cancel

diag matrices commute

- $\mathrm{Tr}([A,B]^2) = \mathrm{Tr}((AB - BA)^2) = \mathrm{Tr}(ABAB - 2ABBA + BABA)$

$$\boxed{\mathrm{Tr}\left[dH\,dH^\dagger\right] = \mathrm{Tr}\left[d\Lambda^2 + 2(S^\dagger dS)\Lambda(S^\dagger dS)\Lambda - 2\Lambda^2(S^\dagger dS)^2\right]}$$

def $\quad dt = S^\dagger dS \quad , dt^\dagger = -dt \quad$ anti hermitian ($\beta = 1$ anti symm)

Write out

$$= \sum_{i=1}^{N} d\lambda_i^2 + 2\sum_{i,j=1}^{N}\big(dt_{ij}\lambda_j\,\underset{-dt_{ij}^*}{dt_{ji}}\lambda_i - \lambda_i^2\,dt_{ij}\,\underset{-dt_{ij}^*}{dt_{ji}}\big)$$

↑ $i=j$ gives 0

$$= \sum_{i=1}^{N} d\lambda_i^2 + 2\sum_{i<j}\overbrace{\left(\lambda_i^2 + \lambda_j^2 - \lambda_i\lambda_j - \lambda_j\lambda_i\right)}^{(\lambda_i - \lambda_j)^2}\,dt_{ij}\,dt_{ij}^*$$

↑ indep. d.o.f. ↑ ↑

check

$\beta=1$: $S=O$, $dH = O^T dO$ $O \in O(N)$

indep var lhs dH_{ij} $i \leq j$ $\frac{N(N+1)}{2}$

rhs $d\lambda_i$ N , dt_{ij} $i<j$ $\frac{N(N-1)}{2}$, $dt_{ij} dt_{ji} = -(dt_{ij})^2$

$$\Rightarrow \mathrm{Tr}[dH dH^T] = \sum_{i=1}^{N} d\lambda_i + 2 \sum_{i<j} \underbrace{(\lambda_i^2 + \lambda_j^2 - \lambda_j \lambda_i - \lambda_i \lambda_j)}_{(\lambda_i - \lambda_j)^2} dt_{ij}^2$$

diagonal in indep. variables !

$$\Rightarrow G^{\beta=1}_{\frac{N(N+1)}{2} \times \frac{N(N+1)}{2}} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 2(\lambda_1-\lambda_2)^2 & & & \\ & & & & 2(\lambda_1-\lambda_3)^2 & & \\ & & & & & \ddots & \\ & & & & & & 2(\lambda_{N-1}-\lambda_N)^2 \end{pmatrix}$$

$$\underbrace{\qquad}_{N} \quad \underbrace{\qquad\qquad}_{\frac{N(N-1)}{2}}$$

$$\Rightarrow \sqrt{\det G} = 2^{\frac{N(N-1)}{4}} \prod_{i<j}^{N} |\lambda_i - \lambda_j|^1$$ Jacobian $\beta=1$

$$\Delta_N(\{\lambda\}) = \prod_{i>j}^{N} (\lambda_i - \lambda_j)$$ Vandermonde det

alternative: from $dH = S(d\Lambda + [\ ])S^{\dagger}$ compute

$$\frac{\partial\, dH_{i\leq j}}{\partial(d\lambda_i, dt_{i<j})}$$

[→ Mehta ch. 3]

section 3.3

$\underline{\beta = 2}$ $\quad S = U \quad dt = U^\dagger dU \quad U \in U(N) \quad H = U \Lambda U^\dagger$

- $H = H^\dagger$ complex hermitian $\Rightarrow N + 2\frac{(N-1)N}{2} = N^2$ indep var rhs

$$dH_{ij}, \; i \leq j \begin{Bmatrix} i=j \in \mathbb{R} \\ i<j \in \mathbb{C} \end{Bmatrix}$$

- $\Lambda = \operatorname{diag} \lambda_i, \lambda_i \in \mathbb{R} \Rightarrow N$ dof.
- $UU^\dagger = U^\dagger U = 1$ unitary $\quad \sum_j U_{ij} U^*_{kj} = \delta_{ik} =$

$2N^2 - N^2$ cond $= N^2$ dof

$\Rightarrow$ rhs has $N^2 + N$ dof ↯ ?

wayout: take $\underset{= -dt^\dagger}{dt}$ anti-hermitian $\Rightarrow N^2 - N$ as indep d.o.f.

(Why: $H = U \Lambda U^\dagger$ is invariant under $U \to U\hat{U}$

with $\hat{U} = \begin{pmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_N} \end{pmatrix}$, ev λ_i unchanged

so we only need $U \in U(N)/U(1)^N$ coset space "Stiefel manifold")

$$\Rightarrow \operatorname{Tr}(dH\,dH^\dagger) = \sum_{i=1}^{N} d\lambda_i^2 + 2 \sum_{i<j} (\lambda_i^2 + \lambda_j^2 - \lambda_i\lambda_j - \lambda_j\lambda_i)\, dt_{ij}\, dt^*_{ij}$$

again diag in indep d.o.f.

$$G = \begin{pmatrix} \underbrace{1 \ddots 1}_{N} & & & \\ & \underbrace{2(\lambda_1-\lambda_2)^2 \quad 2(\lambda_1-\lambda_2)^2 \quad \ddots}_{N^2-N} & & \end{pmatrix} \begin{matrix} \leftarrow \operatorname{Re} dt_{ij} \\ \leftarrow \operatorname{Im} dt_{ij} \end{matrix}$$

– diagonal in indep. variables

$$\Rightarrow \sqrt{\det G} = 2^{\frac{N(N-1)}{2}} \prod_{i<j} |\lambda_i - \lambda_j|^{\overset{\swarrow \beta=2}{2}} \sim |\Delta_N(\lambda)|^2 \text{ Jacobian}$$

$\underline{\beta = 4}$: dH (and H) has $2N^2-N$ d.o.f.

$$\left.\begin{aligned} d\Lambda &= \mathrm{diag}\left(\begin{pmatrix} d\theta_j & 0 \\ 0 & d\theta_j \end{pmatrix}\right) && N \text{ d.o.f.} \\ dt_{i<j} &= \text{need } \; 2N^2 - 2N = 2N(N-1) = 4\,\frac{N(N-1)}{2} \end{aligned}\right\} \checkmark$$

4 per upper triang.
2×2 matrix

$\Rightarrow$ we get $(\lambda_1-\lambda_2)^2$ 4 times from $dt_{12}^{(j=0,1,2,3)}$

$$\Rightarrow \quad \text{Jacobian} = \underline{2^{N(N-1)} \prod_{i<j}^{N} |\lambda_i - \lambda_j|^{4 \leftarrow \beta}}$$

- $\mathrm{Tr}\, V(H)$ is invariant under rotations

$$\Rightarrow \quad Z_N^\beta = \int dH\, e^{-\mathrm{Tr} V(H)}$$

$\int dH \equiv \prod \int_{\mathbb{R}} \frac{dH_{ij}}{\sqrt{2\pi}}$
indep matrixelle
flat Haar

same Jacobian
$$\boxed{Z_N^\beta = 2^{\frac{\beta N(N-1)}{4}}\, C_N^{(\beta)} \underbrace{\left(\prod_{i=1}^{N} \int_{\mathbb{R}} d\lambda_i\right) e^{-\sum_{i=1}^{N} V(\lambda_i)} \, |\Delta_N(\{\lambda\})|^\beta}_{}}$$

these will be computed later

$$\text{with} \quad C_N^{(\beta)} = \begin{cases} \beta = 1 & \int dO \quad \text{Volume of } O(N) \\ \beta = 2 & \int dU \quad \text{vol of } U(N)/U(1)^N \\ \beta = 4 & \int dB \quad \text{vol of } Sp(N)/U(1)^N \end{cases}$$

- these constants will drop out when we compute invariant expectation values, e.g. of $\mathrm{Tr}(H^4)$

$$\mathbb{E}\left(\mathrm{Tr}(H^4)\right) \equiv \frac{1}{Z_N^\beta} \int dH\, \mathrm{Tr}(H^4)\, e^{-\mathrm{Tr} V(H)}$$

* we will compute Z_N^β and all eigenvalue density correlation functions ("marginals") using Orthogonal polynomials!

⑫

8. Orthogonal Polynomials of real variables (see also Livan, Novaes, Vivo)

- Let $\omega(\lambda) > 0$ be a measurable weight on $D \subseteq \mathbb{R}$ such that for all

moments $\boxed{m_k = \int d\lambda\, \omega(\lambda)\, \lambda^k} < \infty$

- Consider $\tilde{P}_k(\lambda) = \lambda^k + d_k \lambda^{k-1} + \ldots$ monic polynomial of degree k

It is called orthogonal polynomial (OP)

if it satisfies orthogonality (=scalar product) $\boxed{\int_D d\lambda\, \omega(\lambda)\, \tilde{P}_k(\lambda)\, \tilde{P}_\ell(\lambda) = \delta_{k\ell}\, h_k} \quad \forall k, \ell \in \mathbb{N}$

$h_k = \|\tilde{P}_\ell\|^2$

- the $h_k > 0$ are the squared norms

[Literature W. van Assche, OP in the complex plane and on the real line, Fields Inst. Commun, Vol 14 (1997) 211]

Note $\tilde{P}_k(\lambda)$ can be constructed recursively using

Gram-Schmidt

$$\tilde{P}_k(\lambda) = \frac{1}{\det[m_{ij}]} \det \begin{vmatrix} m_0 & m_1 & & m_k \\ m_1 & m_2 & & \\ \vdots & & & \\ m_{k-1} & m_k & & m_{2k-1} \\ 1 & \lambda & \ldots & \lambda^k \end{vmatrix}_{(k+1)\times(k+1)}$$

($\Rightarrow$ normalisab is $\det(m_{i+j})_{0,\, k-1}$)

proof: $\int d\lambda\, \lambda^\ell \tilde{P}_k(\lambda)\, \omega(\lambda) = 0 \;\forall \ell < k$

- the OP satisfy a 3-step recurrence relation

monic: $\boxed{\lambda \tilde{P}_k(\lambda) = \tilde{P}_{k+1}(\lambda) + a_k \tilde{P}_k(\lambda) + b_k \hat{P}_{k-1}(\lambda)}$

proof: consider as $\tilde{P}$ basis $\lambda \tilde{P}_k(\lambda) = \sum_{\ell=0}^{k+1} \alpha_k^\ell \tilde{P}_\ell(\lambda)$, $\int d\lambda\, \omega\, \lambda \tilde{P}_k \tilde{P}_j$

- the coefficients a_k, b_k can be expressed in terms of norms h_k and second coeff d_k (see e.g. Abramowitz - Stegun, chapter 22, NIST chapter OP: 18)
- $\tilde{P}_k'(\lambda)$ is also expressible, later