Generators of symmetry transformations

consider the previous examples, but as <u>infinitesimal trafos</u>:

<u>time</u>: $f(t) \to f(t+\tau) = f(t) + \tau \dot{f}(t) + O(\tau^2) = (1 + \tau \partial_t) f(t) + O(\tau^2)$

$\uparrow$ time transl. generator

<u>space</u>: $f(\vec{q}) \to f(\vec{q}+\vec{a}) = (1 + \underbrace{\vec{a}\,\vec{\nabla}}_{= a_i \partial_i}) f(\vec{q}) + O(|\vec{a}|^2)$

Nabla

$a_i \partial_i$ directional derivative, $\partial_i = \frac{\partial}{\partial q_i}$

<u>rotations</u> $f(\vec{q}) \to f(\vec{q} + \delta\alpha \vec{n} \times \vec{q}) = (1 + \delta\alpha (\vec{n} \times \vec{q}) \cdot \vec{\nabla}) f(\vec{q}) + O(\delta\alpha^2)$

$$= (1 + \underbrace{\delta\alpha\, \vec{n} \cdot (\vec{q} \times \vec{\nabla})}_{= \delta\alpha\, \varepsilon_{ijk} n_i q_j \partial_k}) f(\vec{q}) + \ldots$$

- What happens if we enchain several trafos, do they commute?

* <u>translations</u> (= repetition of partial derivatives) ∂_j generate

commute (for smooth functions) $\boxed{[\partial_j, \partial_k] = 0}$ $(\vec{a}_1 + \vec{a}_2 = \vec{a}_2 + \vec{a}_1)$

* <u>rotations</u> do not commute $\Rightarrow$ combi of rotations

generators $M_{ij} = q_i \partial_j - q_j \partial_i = -M_{ji}$ antisym ($\exists$ 3 indep comp in $\mathbb{R}^3$)

$\Rightarrow$ $\boxed{[M_{ij}, M_{kl}] = \delta_{jk} M_{il} - \delta_{jl} M_{ik} - \delta_{ik} M_{jl} + \delta_{il} M_{jk}}$ check in ex 1.2

* <u>rotations & translations</u> $\Rightarrow$ combi of translation

$\boxed{[\partial_k, M_{ij}] = \delta_{ki} \partial_j - \delta_{kj} \partial_i}$ check: ex 1.2

$\Rightarrow$ translations & rotations together close among each other
"form an algebra that generates the group of trafos"

Q: Are there further symmetry transformations under continuous coordinate transformations?

- in relativistically invariant theories yes:

∂_t & $\partial_j \to \partial_\mu$, $M_{ij} \to M_{\mu\nu}$ Lorenz trafos & rotations

* spacetime indep. (global) coordinate trafo $\to$ special relativity (flat space)
* spacetime dependent (local) " " $\to$ general relativity (curved space, particles interact through gravity)

* sometimes conformal symmetry

$\to$ under certain conditions these are <u>all</u> possible symmetries of a "reasonable" (quantum field) theory under <u>coordinate trafos</u>

* in order to enlarge the number of possible symmetries (e.g. by inner symmetries) we now consider <u>fields</u>.

$\to$ we need to generalise the Lagrange formalism to fields, in order to prove the most general form of

<u>Noether's* theorem</u>:

$\exists$ a continuous symmetry $\Longleftrightarrow$ $\exists$ a conserved charge & conserved current
(coordinate and/or inner trafo)

* Emmy Noether (1882 – 1935)

* fields (relativistic) [P. Ramond Field Theory, 1990 Addison Wesley]

replace coordinate $t \to x^\mu = (x^0, x^i)$ $i = 1,2,3$, space time coord

degree of freedom $q(t) \to \phi(x^\mu)$ field

Skalar product $x_\mu x^\mu = x_\mu x_\nu \eta^{\mu\nu} = x^\mu x^\nu \eta_{\mu\nu}$,

Metric $\eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

derivative $\partial_\mu = \frac{\partial}{\partial x^\mu}$, $\partial^\mu = \frac{\partial}{\partial x_\mu}$, $\Rightarrow \partial_\mu x^\nu = \delta_\mu^{\ \nu} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

Lagrangedensity $\mathcal{L}(\phi, \partial_\mu \phi(x_\mu))$ in general not explicitly x^μ-dep.

Lagrangefunction $L[\phi, \partial_\mu \phi] = \int_V d^3x\, \mathcal{L}(\phi, \partial_\mu \phi)$

action $S[\phi] = \int_{t_1}^{t_2} dt\, L[\phi, \partial_\mu \phi] = \int_\Omega d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$, $\Omega: [t_1, t_2] \times V$ 4-Volume

Euler-Lagrange eq. $\boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0}$

example free Skalarfield $\mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2$

eq. of motion $(\partial_\mu \partial^\mu + m^2)\phi = 0$ Klein-Gordon

Derivation of Noether's Theorem:

general Symmetry trafo

Coordinates $x^\mu \to x^{\mu'} = x^\mu + \delta x^\mu$, $\delta x^\mu = X_i^\mu(x)\, \delta\omega^i$

fields $a = 1..N$ $\phi^a(x) \to \phi^{a'}(x') = \phi^a(x) + \delta\phi^a(x)$, $\delta\phi^a(x) = \Phi_i^{\ a}(x)\, \delta\omega^i$

note the infinitesimal parameters $\delta\omega^i$, $i = 1..n$ are x^μ-indep.!

• assumption: these transformations of coordinates and fields are a symmetry of the action

$$\Delta S' = \int_\Delta d^4x' \, \mathcal{L}(\phi^{a\prime}(x'), \partial'_\mu \phi^{a\prime}(x')) = \Delta S + \mathcal{O}(\delta\omega^{i2})$$

for small 4-volumes Δ, where ϕ satisfies the e.o.m. (= is on shell), the vanishing $\mathcal{O}(\delta\omega^i)$ leads to the conserved charge

• Symmetry trafo:

measure: Jacobi-det: $d^4x' = \det\left[\frac{\partial x^{\mu\prime}}{\partial x^\nu}\right] d^4x = \det\left[\delta_\nu^{\ \mu} + \partial_\nu \left(X^\mu_{\ i}(x)\delta\omega^i\right)\right] d^4x$

$\rightarrow$ Taylor expand $\det(\mathbb{1} + xA) = e^{\mathrm{Tr}\log(\mathbb{1}+xA)}$

$= 1 + x \,\mathrm{Tr} A + \mathcal{O}(x^2)$

$$\left(1 + \partial_\mu X^\mu_{\ i}\,\delta\omega^i\right) d^4x + \mathcal{O}(\delta\omega^{i2})$$

– Lagrange density:

$$\mathcal{L}(\phi^{a\prime}(x'), \partial'_\mu \phi^{a\prime}(x')) = \mathcal{L}(\phi^a(x), \partial_\mu \phi^a(x)) + \frac{\partial\mathcal{L}}{\partial\phi^a(x)}\delta\phi^a(x) + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a(x)}\delta(\partial_\mu\phi^a(x)) + \mathcal{O}(\delta\omega^{i2})$$

· Variation of the derivative

$$\begin{aligned}
\delta(\partial_\mu\phi^a) &= \partial'_\mu\phi^{a\prime}(x') - \partial_\mu\phi^a(x) = \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu\phi^{a\prime}(x') - \partial_\mu\phi^a(x) \\
&= \left(\delta_\mu^{\ \nu} - \partial_\mu X^\nu_{\ i}\delta\omega^i\right)\partial_\nu\phi^{a\prime}(x') - \partial_\mu\phi^a(x) \\
&= \partial_\mu\left(\phi^{a\prime}(x') - \phi^a(x)\right) - \left(\partial_\mu X^\nu_{\ i}\right)\delta\omega^i\partial_\nu\phi^a(x) + \mathcal{O}(\delta\omega^{i2}) \\
&= \left(\partial_\mu\Phi^a_{\ i} - \left(\partial_\mu X^\nu_{\ i}\right)\partial_\nu\phi^a(x)\right)\delta\omega^i + \mathcal{O}(\delta\omega^{i2})
\end{aligned}$$

$$\Rightarrow \Delta S' = \Delta S + \int_\Delta dx^4 \left\{ \delta_\nu^\mu (\partial_\mu \bar{X}^\nu_i) \mathcal{L} + \underbrace{\frac{\partial \mathcal{L}}{\partial \phi^a}}_{\overset{\text{E.L.}}{=} \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a}} \bar{\Phi}^a_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a} (\partial_\mu \bar{\Phi}^a_i - (\partial_\mu \bar{X}^\nu_i) \partial_\nu \phi^a) \right\} \delta\omega^i + \mathcal{O}(\delta\omega^{i2})$$

goal: Write $\{\}$ as a total derivative → is already

rest: use $\partial_\mu \left(\delta_\nu^\mu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \partial_\nu \phi^a \right) = 0$

$$\Rightarrow \Delta S' = \Delta S + \int_\Delta dx^4 \{ \partial_\mu j^\mu_i \} \delta\omega^i + \mathcal{O}(\delta\omega^{i2})$$

Symmetry: divergence of current j^μ_i vanishes $\forall i = 1 \dots n$

Conserved current $$j^\mu_i = \underbrace{\left(\mathcal{L} \delta^\mu_\nu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \partial_\nu \phi^a\right)}_{\equiv -T^\mu{}_\nu \text{ Energy-momentum tensor}} \bar{X}^\nu_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \bar{\Phi}^a_i$$

Conserved charge $Q_i = \int_V dx^3 j^0_i$ da $\partial^0 Q_i = 0$

- we can read off the symmetry trafos directly from the conserved current: Noether's Thm is an equivalence!

* in how far do inner symmetries (e.g. spin SU(2)) and coordinate trafos mix, or allow a description through a unified theory?