The Coleman-Mandula Theorem [M.F. Sohnius, Phys. Rep. 128 (1998) 39-204]

Assumptions: Given a local[1], relativistic QFT with a non-trivial S-matrix[2] and with a unique massless ground state[3] as well as a finite gap to the 1st excited one-particle state:

* Then the maximal symmetry under coordinate trafos is given by the Poincaré-group with algebra

$$\boxed{\begin{aligned}&[P_\mu, P_\nu] = 0, \quad [P_\mu, M_{\rho\sigma}] = \eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho \\ &[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\rho} M_{\nu\sigma} + \eta_{\mu\sigma} M_{\nu\rho}\end{aligned}}$$

where P_ν $(= i\partial_\nu)$ is the 4-momentum
$M_{\mu\nu}$ are the angular momentum and boost (Lorentz trafo) generators

* all possible inner (bosonic) symmetries are described by Lie-groups (= "continuous") (e.g. SU(2)) with algebra
$[B_r, B_s] = i f_{rs}{}^t B_t$ which do not mix with the Poincaré algebra (= are an inner product with it)

$$[B_r, P_\mu] = 0 = [B_r, M_{\mu\nu}]$$

That implies that these 2 symmetry classes GR: gravitation ↔ SM: SU(3)×SU(2)×U(1) cannot be unified into a larger Symmetry group (in a non-trivial way) containing both as a subgroup!

where 2) S-matrix means

$S_{fi} = \langle \text{final} | \text{initial} \rangle$ is trivial if always $\sim \delta_{fi}$

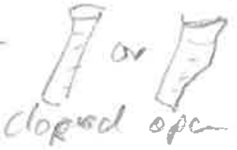
in general $\langle f | i \rangle$ are represented by plane waves at $t = \pm\infty$

with time evolution (in QM) $|t, \text{state}\rangle = e^{-(t-t_0)\hat{H}} |t_0, \text{state}\rangle$

in the Schrödinger picture. One- or multiparticle states are obtained by applying creation operators to the vacuum

exceptions:

* to 1) locality: this concept is given up e.g. in string theory with a smallest spacial resolution
  -> allows a unified theory



* to 3) for a degenerate vacuum (all one particle states are massless) conformal sym. is allowed in addition to Poincaré as a coordinate trafo.
  But these 2 still don't mix with the inner sym!

* Supersymmetry: the bosonic character of the generators is given up -> graded groups (Lie ~)

fermionic Susy generators $Q_{\alpha i}, \; Q^{+}_{\dot{\alpha} i}$ $i = 1, \dots, \mathcal{N} = \#$ of supersymmetries

superalgebra:

$$\{ Q_{\alpha i}, Q^{+j}_{\dot{\beta}} \} = 2 \delta_i^j (\sigma^\mu)_{\alpha\dot{\beta}} P_\mu$$

$$\{ Q_{\alpha i}, Q_{\beta j} \} = 2 \varepsilon_{\alpha\beta} a^r_{ij} B_r \qquad \text{etc.}$$

=> mix Fermions & Bosons,

mediated through the Q's ~~now~~ P_μ and B_r (and others) can mix -> unification is possible

Conformal Symmetry [Lit: P. Ginsparg, hep-th/9108028, chapter 1]

- example for a non-trivial coordinate trafo
- generalises Poincaré-trafo (translate, rotate, boost) to angle preserving ones
- particularly important in 2 dim (1+1): $\exists\ \infty$ many generators
  $\longrightarrow$ complete integrability, conformal field theory (CFT)
- important in models of gravity and the description of critical phenomena: close to phase transitions (2nd or higher order) the correlation length diverges $\Rightarrow$ theory has conformal sym (no scales)

* general infinitesimal coordinate trafo

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x) + \mathcal{O}(\epsilon^2) \qquad \mu = 0, 1, \ldots, d-1$$

in d space-time dim

(e.g. $\epsilon^\mu(x) = a^\mu$ const are translations)

* tensor transformation law for tensor of n-th order

$$\boxed{\phi_{\mu_1 \ldots \mu_n}(x) \to \phi'_{\mu_1 \ldots \mu_n}(x') = \frac{\partial x^{\nu_1}}{\partial x'^{\mu_1}} \cdots \frac{\partial x^{\nu_n}}{\partial x'^{\mu_n}} \phi_{\nu_1 \ldots \nu_n}(x)}$$

e.g. • scalar $\phi(x) \to \phi'(x') = \phi(x)$ is invariant

Cherenko inner sym: p. 8 with $\Phi \equiv \phi$

• Vector $A_\mu(x) \to A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x)$ (this is a coordinate, not a gauge trafo)

tensor of 2nd order

e.g. • metric $g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x) \equiv g_{\mu\nu}(x) + \delta g_{\mu\nu}(x)$

it holds $\frac{\partial x^\rho}{\partial x'^\mu} = \frac{\partial}{\partial x'^\mu}\left(x'^\rho - \epsilon^\rho(x)\right) = \delta^\rho_\mu - \partial'_\mu \epsilon^\rho(x) + \mathcal{O}(\epsilon^2)$ drop as of higher order

$$\Rightarrow \boxed{\delta g_{\mu\nu}(x) = -\left(g_{\rho\nu}(x)\, \partial_\mu + g_{\mu\rho}(x)\, \partial_\nu\right) \epsilon^\rho(x)} + \mathcal{O}(\epsilon^2)$$

general infinites. coord trafo of metric

assume $g_{\mu\nu}(x) = g_{\nu\mu}(x)$

example scalar product among vectors

$$v \cdot w = v^\mu w^\nu g_{\mu\nu} = v^\mu w_\mu$$

is a scalar and invariant under coordinate trafos.

e.g. translations $\epsilon^\mu_{(x)} = a^\mu$ const $\Rightarrow \frac{\partial x^\rho}{\partial x'^\nu} = \delta^\rho_\nu + \mathcal{O}(\epsilon^2)$

but also under rotations or Lorentz trafos

- Consider a specific class of trafos that is characterised through its action on the metric:

Weyl-transformation $g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x)\, g_{\mu\nu}(x)$

or infinitesimally $\Omega(x) = 1 + \omega(x)$, $|\omega(x)| \ll 1$

so $\boxed{\delta g_{\mu\nu}(x) = \omega(x)\, g_{\mu\nu}(x)}$ i.e. for $\omega(x) > 0$ (< 0) growing (shrinking)

- **Weyl-trafos preserve the angle:**

$$v \cdot w = \|v\| \cdot \|w\| \cdot \cos(\theta)$$

norm $\|v\|^2 = v \cdot v$

for $v^\mu, w^\mu \neq \vec{0}$

$\Rightarrow$ Weyl gives $v \cdot v = v^\mu v^\nu g_{\mu\nu} \to \Omega(x)\, v^\mu v^\nu g_{\mu\nu} = \Omega(x) \|v\|^2$

ditto $v \cdot w \to \Omega\, v \cdot w$ and so finally

$$\cos\theta = \frac{v \cdot w}{\|v\|\,\|w\|} \quad \text{is invariant}$$

Def: A conformal trafo is any coordinate trafo, that acts on the metric as a Weyl trafo (incl. scale factor $\Omega(x) \equiv 1$)

this means $\epsilon^\mu(x)$ satisfies $\delta g_{\mu\nu}(x) = -(g_{\rho\nu}\partial_\mu + g_{\rho\mu}\partial_\nu)\epsilon^\rho(x) \overset{!}{=} \omega(x)\, g_{\mu\nu}(x)$

- simplify with inverse metric $g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$, $a^\mu = g^{\mu\nu} a_\nu$

• let us express $\omega(x)$ as a function of $\epsilon^\mu(x)$, and then obtain an equation that determines $\epsilon^\mu(x)$ in order to qualify as conformal

$$g^{\mu\alpha} g^{\nu\beta} \cdot \delta g_{\mu\nu} : \quad -\left(\delta^\beta_\rho \partial^\alpha + \delta^\alpha_\rho \partial^\beta\right) \epsilon^\rho(x) = g^{\alpha\beta} \omega(x)$$

$$\Longleftrightarrow \quad \underline{-\partial^\alpha \epsilon^\beta(x) - \partial^\beta \epsilon^\alpha(x) = g^{\alpha\beta} \omega(x)}$$

use that $d = \mathrm{Tr}\, \mathbb{1} = \mathrm{Tr}\, g g^{-1} = g_{\alpha\beta} g^{\beta\alpha}$

$\Rightarrow \cdot g_{\beta\alpha}$ gives $\quad \underline{-2 \partial_\beta \epsilon^\beta(x) = \omega(x) \cdot d}$

$$\Rightarrow \quad \boxed{\partial^\alpha \epsilon^\beta(x) + \partial^\beta \epsilon^\alpha(x) = \frac{2}{d} \left(\partial_\rho \epsilon^\rho(x)\right) g^{\alpha\beta}} \qquad (*)$$

* <u>all infinitesimal trafos</u> $\epsilon^\alpha(x)$ <u>that solve this eq. are conformal trafos</u>!

* distinguish $d=1$ (trivial, why?), $d=2$ and $d>2$:

<u>$d>2$</u>: multiply $(*)$ with $\partial_\sigma \partial_\alpha$ & assume that all ∂_ρ commute

$$\Rightarrow \quad \underline{\partial_\sigma \partial_\alpha \partial^\alpha \epsilon^\beta(x) + \left(1 - \frac{2}{d}\right) \partial_\sigma \partial^\beta \partial_\rho \epsilon^\rho(x) = 0}$$

one can show that this implies $\partial^\mu \partial^\nu \partial^\rho \epsilon^\sigma(x) = 0$

$\Rightarrow \epsilon^\mu(x)$ is at most quadratic in x^ρ

• an <u>Ansatz</u>

$$\epsilon^\mu(x) = A^\mu + B^\mu{}_\nu x^\nu + C^\mu{}_{\nu\rho} x^\nu x^\rho$$

where the relation between the coefficients A, B, C and known as well as new trafo's is determined in the exercises.