Result: The conformal transformations (group) in $d>2$

1) translations $x^\mu \to x^\mu + A^\mu$
2) rotations and boosts $x^\mu \to x^\mu + \omega^\mu{}_\nu x^\nu$
with $\omega_{\mu\nu} = -\omega_{\nu\mu}$ (antisym part of B)

} Poincaré

3) scale trafos $x^\mu \to x^\mu + b x^\mu$ (sym. part of B)

4) special conformal trafo $x^\mu \to x^\mu + c^\mu x\cdot x - 2x^\mu c\cdot x$
$$= x^\mu + c^\mu x_\nu x^\nu - 2x^\mu c_\nu x^\nu$$

- these are all infinitesimal conformal trafos in $d>2$.
- the corresponding set of generators is (they form the "algebra" of the conformal group)

1) $P_\mu = \partial_\mu$
2) $M_{\mu\nu} = \frac{1}{2}(x_\mu \partial_\nu - x_\nu \partial_\mu)$
3) $D = x^\mu \partial_\mu$
4) $K_\mu = x\cdot x\, \partial_\mu - 2 x_\mu x^\nu \partial_\nu = x_g x^g \partial_\mu - 2 x_\mu x^\nu \partial_\nu$

- in $d=2$ the 2 coordinate x^0, x^1 can be mapped to complex coordinates z and $\bar z$. The corresponding trafos are then all meromorphic and all anti-meromorphic trafos, and thus there are ∞ many generators (→ exercise)

$$L_n = -z^{n+1}\partial_z, \quad \bar L_n = -\bar z^{n+1}\partial_{\bar z} \qquad n\in\mathbb{Z}$$

they form the classical Virasoro algebra

$$\boxed{\begin{array}{l}[L_n, L_m] = (n-m) L_{n+m} \\ [L_n, \bar L_m] = 0\end{array}}$$

ditto for $\bar L$'s

Part V: Inner Symmetries & Group Theory [Lit: book by H.F. Jones]

Definitions, Examples and discrete groups

Def Group (and semi-group)

A group G is a set of elements $\{a, b, \ldots\}$ together with a law of composition („multiplication"), that assigns to every ordered pair $a, b \in G$ an element $a \cdot b \in G$. It has to satisfy

(G0 (closure): $\forall a, b \in G \Rightarrow a \cdot b \in G$

G1 (associative law) $\forall a, b, c \in G$ it holds $a(b \cdot c) = (a \cdot b) \cdot c$) these define a semi-group

G2 (existence of unit element): $\exists e \in G$ s.th. $\forall a \in G: a \cdot e = a = e \cdot a$

G3 (existence of inverse): $\forall a \in G \; \exists a^{-1} \in G$ s.th. $a a^{-1} = e = a^{-1} a$

- in general $a \cdot b \neq b \cdot a$, so multiplication is not commutative

Def A group G is called Abelian if $\forall a, b \in G: ab = b \cdot a$
(else it is called non~)

Exercise: e is unique, , left and right inverses are unique & equal

Def For groups with finitely many elements the order of that group is the number of elements

(one may also define an order for groups with ∞ many elements and relate them)

Examples

1.) C_2 : $\{e, a\}$ with multiplication $a \cdot a = e$

- has order 2 and is abelian

the composition can be represented by a

multiplication table (Latin square)

$g_1 \backslash g_2$	e	a
e	e	a
a	a	e

$g_1, g_2 \in G$ run through all elements

- a possible representation of C_2 on $\mathbb{C}$ is given by $\{1, e^{i\pi}\}$

with standard multiplication on $\mathbb{C}$: $e^{i\pi} \cdot e^{i\pi} = e^{2\pi i} = 1$

($e^{i\varphi} = \cos\varphi + i\sin\varphi$ Euler)

2.) $\mathbb{Z}_n$ defined by the addition modulo n of the integers $\{0, 1, 2, \ldots, n-1\}$

It has order n and is abelian : $n_1 + n_2 = n_2 + n_1$

An example is $n = 2$

	0	1
0	0	1
1	1	0

$\mathbb{Z}_2$ is isomorphic to C_2 (denoted by $\mathbb{Z}_2 \cong C_2$),

meaning there exists a unique, invertible map among the 2 groups ($0 \leftrightarrow e$, $a \leftrightarrow 1$) ($\mathbb{Z}_3$ plays an important role in QCD: $SU(3)$)

3.) S_n: the permutations of n objects. S_n has order $n!$

e.g. for $n=2$: $e: \begin{pmatrix} 1 \to 1 \\ 2 \to 2 \end{pmatrix}$, $a: \begin{pmatrix} 1 \to 2 \\ 2 \to 1 \end{pmatrix}$, and $S_2 \cong \mathbb{Z}_2$

identity　　　transposition

4.) Multiplication among the integers $\mathbb{Z}$ does not form a group, since the inverse of n is $\frac{1}{n}$ which is rational and not in $\mathbb{Z}$

Some simple point groups (discrete groups)

- the $\boxed{\text{cyclic group } C_n}$: defined by the symmetry group of rotations of a regular polygon with n directed equal sides

e.g.

the elements of C_n are $e, c, c^2, \dots, c^{n-1}$ with $c^n = e$, so it has order n. Here c represents a rotation around the origin in $\mathbb{R}^2$ by $\frac{2\pi}{n}$ in positive (= counter clockwise) sense.

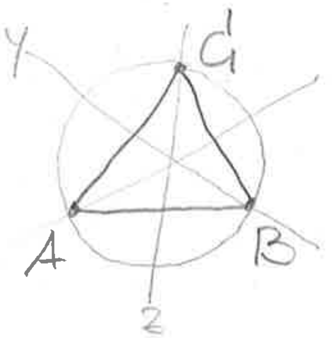
* a representation of C_n on the unit circle is given for $n=3$ by $c = e^{\frac{2\pi i}{3}}$, so $(1, c, c^2) \leftrightarrow (1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}})$

* C_n is abelian, $c^{n_1} c^{n_2} = c^{n_2} c^{n_1} = c^{n_1+n_2}$ and $C_n \cong \mathbb{Z}_n$ with the map $c^{n_1} \leftrightarrow n_1$ for $n_1 = 0, 1, 2, \dots, n-1$

* why can C_3 and S_3 not be isomorphic ?

- the $\boxed{\text{Dihedral group } D_n}$: like C_n, but with undirected sides

e.g. $n=3$



$\Rightarrow$ in addition to the rotations from C_3 in the plane $\mathbb{R}^2$, there exist 3 further rotations in $\mathbb{R}^3$ (or mirrorings in $\mathbb{R}^2$) by π around AX, BY, CZ that preserve Δ

element b_1, b_2, b_3 $\Rightarrow b_1^2 = b_2^2 = b_3^2 = e$

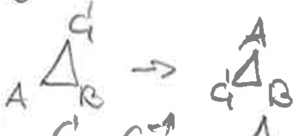
$b_1 \equiv b$

$\Rightarrow D_3$ has order 6.

in general D_n has order $2n$

* can do the same operation in different ways:

e.g. $b_2 = \underset{3rd}{c}\ \underset{2nd}{b_1}\ \underset{1st}{c^{-1}}$: b_2 :



exchange A and C

($\Leftrightarrow b_2 c = c b_1 \neq b_1 c$) or $c b_1 c^{-1}$