- these relations among b_2 and b_1 (and more) lead to so called <u>conjugacy classes</u>

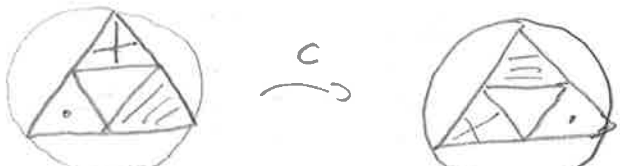
<u>Def</u> A <u>subgroup</u> H of the group G is a subset of the elements of G such that H itself is a group. (c)

<u>examples</u>: C_3 is a subgroup of D_3

- $\{e\}$ is always a subgroup of G, as well as G itself.
- a subgroup H of G with $\{e\} \subsetneq H \subsetneq G$ is called a <u>proper subgroup</u>

* C_n and D_n are not only rotational symmetries of a single regular polygon (n-gon), but also of a <u>lattice</u> of the same polygon (either as a finite or infinite lattice), hence their importance in crystals: (→ exercises)

eg. <u>n = 3</u>



same lattice

- the $\boxed{\text{Permutation group } S_n:}$

* It's an important symmetry group for many particle systems with indistinguishable particles

any permutation $\underline{P}$ of n objects with indices $i = 1, 2, \ldots, n$ can be written as ⟶

$$\underline{P} \begin{pmatrix} 1 & 2 & 3 \ldots & n \\ p_1 & p_2 & p_3 & p_n \end{pmatrix}$$

or $\pi(1)\ \pi(2) \ldots$

where every index $p_{i=1 \ldots n}$ appears once and only once

* the ordering of the columns doesn't matter.

e.g. for $n=3$ $\quad \underline{P} = \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} = \begin{pmatrix}2&1&3\\3&1&2\end{pmatrix}$

* enchaining 2 permutations can be represented by the following multiplication

- do first $\underline{Q} = \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$ and then $\underline{P} = \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$:

written as $\underline{P}\,\underline{Q} = \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix} = \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$

- compare to

$$\underline{Q}\,\underline{P} = \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} = \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} \neq \underline{P}\,\underline{Q}$$

$\Rightarrow$ S_n is non-Abelian $\left(\text{add} \begin{pmatrix}4&5&\cdots&n\\ \cdots 4&5&&n\end{pmatrix}\right)$

* the identity is $e = \begin{pmatrix}1&2&\cdots&n\\1&2&\cdots&n\end{pmatrix}$, and $\underline{P}^{-1} = \begin{pmatrix}p_1&p_2&\cdots&p_n\\1&2&&n\end{pmatrix}$ is inverse to $\underline{P} = \begin{pmatrix}1&2&&n\\p_1&p_2&\cdots&p_n\end{pmatrix}$

* an alternative representation of S_n is with the help of cycles:

given a permutation Q choose an index, e.g. 1 and follow its permutations until you reach again 1:

- if we run through all n indices we stop,

  e.g. for $n=3$ and Q: $1 \to 3 \to 2 \to 1$, denote by $\underline{Q} = (1\,3\,2\,\cancel{1})$

  this is a cycle

  drop

- if we do not cover all n indices pick one that was not covered, follow its permutations etc.

e.g. for $n=3$ and $\underline{P}$: $1 \to 1$ $\quad (1)$

then $2 \to 3 \to 2$ $\quad (2\,3)$ stop as all 3 covered

denote by $\underline{P} = \cancel{(1)}(2\,3)$.

if it is clear that $n=3$ we can drop all cycles of length 1

* because different cycles don't mix their order is not important
here we product 2 first

so for $n=3$ $\quad P = (1)(23) = (23)(1)$

for $n=5$ $\quad R = (124)(35) = (35)(124)$ $\quad$ etc.

* the above multiplication can now be written as

$$PQ = (23)\circ(132) = (12)(3)$$
$$QP = (123)\circ(23) = (13)(2)$$

* cycles are useful to put the elements of S_n into a certain order

- S_2 : group of order 2 : identity $(1)(2)$ or denoted by $()$
  transposition (12)

$\Rightarrow S_2 = \{(); (12)\}$

and $S_2 \cong C_2$

- S_3 : order $3! = 6$ $\quad$ it is given by the identity, all 2 cycles

and all 3 cycles : $S_3 = \{(); (12); (13); (23); (123); (132)\}$

etc. for all S_n

* it is clear that C_n and D_n are subgroups of S_n, as all rotations can be represented as permutations of the n endpoints of the polygon.

it holds $D_3 \cong S_3$: $\{e, b, bc, bc^2, c, c^2\}$ $\quad$ check!

$\{(), (23), (31), (12), (123), (132)\}$

* because of the different orders $C_n : n$ $\quad D_n : 2n$, $\quad S_n : n!$

it is clear that for $n > 3$ C_n and D_n are proper subgroups

e.g. for $n=4$

is a permut. but not a rot. or a reflection

Cayley's Theorem:

Every finite group G of order n is (isomorphic to) a sub-group of the permutation group S_n

proof: multiplying all elements $\{a_1, \dots, a_n\}$ of G by a fixed element $g \in G$ forms a permutation:

$$(g a_1, g a_2, \dots, g a_n) = (a_{\pi_1}, a_{\pi_2}, \dots, a_{\pi_n})$$

that is $g \longrightarrow \pi(g) = \begin{pmatrix} 1 & 2 & \dots & n \\ \pi_1 & \pi_2 & \dots & \pi_n \end{pmatrix}$

- This operation is indeed a permutation as on the right hand side (r.h.s) every element appears once and only once.
  (assume that not: $\exists j \neq k : g a_j = g a_k$, $g^{-1} \cdot \Rightarrow a_j = a_k$ ↯
  $\Rightarrow$ in every row (column) of the Latin square every element appears exactly once)
- The map $g \to \pi(g)$ is unique as $\pi(g)$ cannot be generated by multiplication with another element
  (assume that it can: $\exists i, \exists g' \in G : g a_i = g' a_i$, $\cdot a_i^{-1} \Rightarrow g = g'$ ↯)
- The map $g \leftrightarrow \pi(g)$ respects the group structure on both sides
  let $g_1, g_2 \in G$, $g_1 \leftrightarrow \pi(g_1)$, $g_2 \leftrightarrow \pi(g_2)$

$\Rightarrow \quad g_1 g_2 \leftrightarrow \pi(g_1)\,\pi(g_2)$

given by $(g_1 g_2 a_1, \dots, g_1 g_2 a_n)$

the elements on the lhs are in G, on the rhs in S_n and they satisfy the group axioms.

The n permutations $\pi(a_1), \dots, \pi(a_n)$ are a subset of all $n!$ permutations in S_n $\Rightarrow$ G is a subgroup of S_n ($n>3$ proper sub-group) with unit element $e \leftrightarrow \pi(e)$ equal to the identity

General Properties of Groups and Maps:

<u>Def</u> Conjugation: Two elements $a, b \in G$ (G group) are called conjugate when there exists an element $g \in G$ s.th. $a = g b g^{-1}$. The element g is called conjugating element.

* In general g depends on a and b, but it doesn't have to be unique

example: in D_3 c and c^2 are conjugate, with conjugating element b: $c = b c^2 b^{-1}$ $\quad (\Leftrightarrow b_2 = c b_1 c^{-1})$

further conjugating elements are b_2 $(\to bc)$ and b_3 $(\to bc^2)$ check!

* Conjugation is only a particular example for a so called equivalence relation. Both concepts are important to classify groups and their sub groups.

<u>Def</u> Equivalence relation $\sim$

For a set S we define an equiv. rel $a \sim b$ for $a, b \in S$.
It has to satisfy:

E1) reflexivity $\quad \forall a \in S \quad a \sim a$

E2) symmetry $\quad \forall a, b \in S$ if $a \sim b \Rightarrow b \sim a$

E3) transitivity $\quad \forall a, b, c \in S$ if $a \sim b$ and $b \sim c \Rightarrow a \sim c$

examples:
- Equiv. rel $a \sim b$ if a&b have the same parents
- a and b having the same nationality does not def. an equiv. rel, why?

- Conjugation is an equivalence relation.

check E1) $a \sim a$ holds due to group property G2
with conjugating element $e = e^{-1}$: $a = e a e^{-1}$

check E2) $a \sim b \Leftrightarrow \exists\, g \in G$ with $a = g b g^{-1}$
$\Rightarrow \exists\, h = g^{-1} \in G$ with $h a h^{-1} = g^{-1} a g = b$ s.th. $b \sim a$

check E3) $a \sim b$ and $b \sim c \Leftrightarrow \exists\, g, h \in G$ with
$a = g b g^{-1}$ and $b = h c h^{-1} \Rightarrow \exists\, j = gh \in G$
s.th. $a = g(\underbrace{h c h^{-1}}_{a})g^{-1} = (gh)\, c\, (gh)^{-1}$ s.th. $a \sim c$
as $(gh)^{-1} = h^{-1} g^{-1}$

Conjugacy classes:

- Every equivalence relation $\sim$ on a finite set S allows to partition the set S into disjoint equivalence classes. Here

Def Equivalence class (a) of an element $a \in S$
is given by $(a) \equiv \{ b \mid b \in S \text{ and } b \sim a \}$

This partitioning works in a similar way as the representation of permutations in terms of cycles:

- we start with an $a \in S$ and construct (a). When $(a) \subsetneq S$ we choose $b \in S \setminus (a)$ and construct (b), etc until $(a) \cup (b) \cup \ldots = S$. It holds that (a) and (b) are disjoint, due to E3) (assume $\exists\, d \in (a)$ and $d \in (b) \Rightarrow d \sim a$ and $d \sim b \Rightarrow a \sim b$ ↯)

Def Conjugacy class (a)
For a group G we define $(a) \equiv \{ b \mid b = g a g^{-1}$ for a fixed $g \in G \}$