Examples

1) for Abelian groups (e.g. C_n) every element $a \in G$ forms its own conjugacy class: $\forall a, g \in G \quad ga = ag$:
$b \in G$ with $b \sim a$: $\exists g$ s.th. $\underline{a = gbg^{-1}} = bgg^{-1} = \underline{b}$
Obviously these classes partition G.

2) For the permutation group S_n the conjugacy classes correspond precisely to cycles (check, exercise)

e.g. for $n=3$ S_3

partition	cycle
(a) (b) (c)	( ) identity
(a) (bc)	(1)(23), (2)(13), (3)(12)
(abc)	(123), (132)

Besides conjugacy classes there exists a further possibility to partition a group G:

<u>Def</u> <u>Coset</u>
Given a group G with subgroup H of order r. The <u>left coset</u> of an element $g \in G$, written as gH, is defined through the set of all elements obtained by left multiplication with all elements $\{h_1, .., h_r\} \in H$: $gH = \{gh_1, gh_2, .., gh_r\}$.
Similarly we can define the <u>right coset</u>.

* different cosets are either identical or disjoint, and provide a partitioning of G:
to prove this we construct the following equivalence relation:

$$a \sim b \iff b \in aH$$

* to see that this is indeed an equiv. rel. we check E1) – E3):

check E1) $a \sim a \Leftrightarrow a \in aH$. As H is a group $e \in H$ and hence $a = ae \in aH$

check E2) $a \sim b \Leftrightarrow b \in aH$, so $\exists h \in H$ with $b = ah \Leftrightarrow bh^{-1} = a$ and as H is a group $h^{-1} \in H$, so $a = bh^{-1} \in bH$: $b \sim a$

check E3) <u>transitivity</u> $a \sim b$ and $b \sim c$: $b \in aH$ and $c \in bH$

$\Rightarrow \exists h, h' \in H$ with $b = ah$, $c = bh' = ahh'$.

Now H is a group and hence $hh' \in H \Rightarrow c \in aH$: $a \sim c$

$\square$

Because $a \sim b \Leftrightarrow b \in aH$ is an equivalence relation it follows that cosets are either disjoint or identical. For a finite group G of order s we can count these cosets:

$$\{g_1 H, g_2 H, \ldots, g_{s'} H\} \equiv G/H$$

This defines the set of all left cosets, with $s' \leq s$ as some cosets can be identical.

<u>Lagrange's Theorem</u>:

Given a group G of order s and a subgroup H of G of order r.

Then r is a divisor of s, that is $\exists n \in \mathbb{N}_+$ s.th. $s = n \cdot r$.

<u>proof</u>:

Every coset gH of G contains r elements as $\{gh_1, \ldots, gh_r\}$ are all different ($h_1, \ldots, h_r$ are the elements of H). All left cosets gH partition the elements of G into disjoint sets (as gH defined an equiv. rel.), an all these cosets contain r elements

$\Rightarrow s = n \cdot r$ where $n \in \mathbb{N}_+$ is the number of disjoint cosets gH.

$\square$

Example: $D_3 = \{e, c, c^2, b, bc, bc^2\}$ of order 6

has subgroups • $\underline{H = \{e, b\}}$ (also $\{e, b_2\}$, $\{e, b_3\}$)
$\simeq C_2$ of order 2

• $C_3 = \{e, c, c^2\}$ of order 3

which all divide 6.

The disjoint cosets of H are

$$\left.\begin{aligned} eH &= \{e, b\} = H \\ cH &= \{c, cb\} \\ c^2H &= \{c^2, c^2b\} \end{aligned}\right\} \text{put together this gives all elements of } D_3$$

(check that the remaining cosets will give the same sets, e.g. $bH = \{b, e\}$)

* a group G with an order p prime cannot have proper subgroups!

Normal Subgroups

• In general subgroups and conjugacy classes are concepts that have little in common (e.g. conjugacy classes do not form a group, lacking the element e which always forms conj. class (e))

• An exception are so-called normal or invariant or self-conjugate subgroups. These are characterised in several equivalent ways:

Def A normal subgroup H of G satisfies: $\boxed{\forall g \in G : gHg^{-1} = H}$ *;

this is equivalent to saying that the left and right cosets of H coincide for all $g \in G$: $\{h_1 g, h_2 g, \dots, h_r g\} = \{gh_1, gh_2, \dots gh_r\}$

in other words: $\forall g \in G \ \forall h_i \in H \ \exists h_j \in H$ s.th. $h_i g = g h_j$

- Thus an equivalent characterisation of $*$ is that all conjugates of H must belong to H: $\boxed{\forall g \in G : h \in H \Rightarrow g h g^{-1} \in H}$

- A third, equivalent definition is that H is a normal subgroup if it is made up of complete conjugacy classes (so that conjugation does not lead out of H).

Example D_3:

C_3 is a normal subgroup as $C_3 = \{e, c, c^2\}$ is made up of the conjugacy classes (e) and $(c) = \{c, c^2\}$ (check)

$C_2 = \{e, b\}$ is not a normal subgroup as $(b) = \{b, bc, bc^2\}$ and so conjugation leads out of C_2.

Quotient Groups

- We have already seen $G/H = \{g_1 H, g_2 H, \dots, g_s H\}$, the set of all left cosets of G wrt H. Given that H is a normal subgroup of G we can show that actually G/H forms a group itself, provided we define the following multiplication on it

Def The product of 2 cosets $g_1 H$ and $g_2 H$, $g_1, g_2 \in G$, H subset of G is defined as $(g_1 H) \cdot (g_2 H) \equiv g_1 g_2 H$
which itself is a coset again, as $g_1 g_2 \in G$ too.

It holds that for H a normal subset of G the set G/H with coset product forms a group. To proove this we check the group axioms:

G0) Closure okas $(g_1H)(g_2H) = \underbrace{g_1g_2}_{\in G} H \in G/H$

G1) associativity: is inherited from multiplication on G:

$$(g_1H)[(g_2H)(g_3H)] = (g_1H)\ (g_2g_3H) = g_1(g_2g_3)H$$
$$\overset{G1)\text{ for }G}{=} (g_1g_2)g_3H = (g_1g_2H)(g_3H) = [(g_1H)(g_2H)]g_3H$$

G2) $\exists$ identity element $E \equiv eH = H$ as

$(eH)(gH) = (eg)H = gH$, same for right multiplication

G3) $\exists$ inverse coset: $g^{-1}H$ is inverse to gH:

$(gH)(g^{-1}H) = (gg^{-1})H = eH = H$

Are we done? No, we have to check that our def is consistent. Why? Because when labeling all elements of $G/H = \{g_1H, g_2H, \dots, g_sH\}$ the elements labeling the cosets are not unique, as some cosets may coinside for $g_1H = g_1'H$ say. For the coset multiplication to be well defined we have to show, that differnt rep. lead to the same product, that is

$$g_1'g_2'H \overset{!}{=} g_1g_2H$$

We start with coset $g_1H = g_1'H$. Because $e\in H$ g_1 and g_1' are both in g_1H, $g_1' \in g_1H \Rightarrow \exists h_1 \in H$ s.th. $g_1' = g_1h_1$

Ditto for g_2 and g_2': $g_2' \in g_2 H \Rightarrow \exists h_2 \in H$ s.th. $g_2' = g_2 h_2$

$\Rightarrow$ we have for the product $(g_1' H)(g_2' H)$

$= g_1' g_2' H = g_1 h_1 g_2 h_2 H$. It holds that $h_2 H = H$

like in the proof for Cayleys Thm, h_2 acting on elements of H will induce a permutation of element of H, and not lead out of H.

$*$ Now we use that H is normal: $g_2 H = H g_2$

$$g_1' g_2' H = g_1 h_1 g_2 H = g_1 h_1 H g_2 \overset{\text{see}}{\underset{\text{above}}{=}} g_1 H g_2 \overset{\text{normal}}{=} g_1 g_2 H$$

Example:

We have seen that $H = C_3 = \{e, c, c^2\}$ is a normal subgroup of D_3

The 2 disjoint cosets of this H are

$$eH = H = \{e, c, c^2\} \equiv E$$

and $bH = \{b, bc, bc^2\} \equiv B$

So we have $D_3/C_3 = \{E, B\}$ forms a group itself, that is isomorphic to C_2: We can check this by explicitly computing its multiplication table:

$$E^2 = (eH)(eH) = e^2 H = H = E$$

$$EB = (eH)(bH) = ebH = bH = B$$

$$B^2 = (bH)(bH) = b^2 H = eH = E,$$

So $D_3/C_3 \cong C_2$. If we interpret the coset group G/H as G "divided" by H, can we also multiply groups, i.e. $D_3 \overset{?}{\cong} C_3 \times C_2$? Such cases exist, but not for D_3!