<u>Def</u>: Let $G$ be a group with subgroups $A$ and $B$. Then $G$ is the <u>direct product</u> written as $G = A \times B$ if it holds

i) all elements of $A$ commute with all elements of $B$ (*)

ii) every element of $G$ can be written in a <u>unique</u> way as $g = ab$:
$\forall g \in G \ \exists!\, a \in A$ and $\exists!\, b \in B$ s.th. $g = ab$

The multiplication on such a direct product is well defined due to i):

$$g_1 g_2 = (a_1 b_1 a_2 b_2) \overset{i)}{=} \underbrace{a_1 a_2}_{\in A} \underbrace{b_1 b_2}_{\in B}$$

so that also the product lies in $G$.

(*) note that $A$ or $B$ are not necessarily Abelian themselves

- <u>It follows that</u>:
- $A$ and $B$ are <u>normal subgroups</u> of $G$:
  $\forall g \in G$: $a_i \in A \Rightarrow g a_i g^{-1} = a b a_i b^{-1} a^{-1} = a b b^{-1} a_i a^{-1} = a a_i a^{-1} \in A$ ✓
  and ditto for $B$
- The quotient group $G/A \simeq B$ and $G/B \simeq A$:
  $A \times B/A$ $\qquad$ $A \times B/B$
  Let $B = \{b_1, b_2, \ldots, b_r\}$. Then all cosets $b_j A$ are distinct
  (If not $\exists b_1, b_2$: $b_1 A = b_2 A$, i.e. $\exists\, b_1 a_i = b_2 a_j$. This contradicts the unique way of writing elements of $g = ab$).
  The multiplication of $b_j A$ follows that of the $b_j$, and we can identify the coset $G/A$ with $B$. Ditto for $G/B \simeq A$.

$\rightarrow$ We can multiply & divide groups in this case $G/A = A \times B/A \simeq B$

<u>Examples</u>: • $D_2$ has 2 subgroups $C_2 = \{e, c\}$, $C_2' = \{e, b\}$, $c^2 = e = b^2$

$= \{e, c, b, cb\}$. Both are normal because of $cb = bc$, $\Rightarrow$ i)
Every element of $D_2$ can be written as a unique comb. of $C_2$ and $C_2'$ elements ii)
$\Rightarrow D_2 = C_2 \times C_2$, $\quad D_2/C_2 = C_2$

alternative, constructive definition of the direct product
also called product group:

Def': Take 2 groups $A$ and $B$ and form their
Cartesian product $A \times B$: this is a group if we def
the following multiplication on it: $a, a' \in A$, $b, b' \in B$
$(a,b) \in A \times B$, $(a',b') \in A \times B$ with
$$(a,b) \cdot (a',b') = (aa', bb') \in A \times B$$
The identity is $(e_A, e_B)$ and the inverse of $(a,b)$ is $(a^{-1}, b^{-1})$
Then $A \times B$ is a product group

* We have seen that $D_3/C_3 = C_2$ p. 22, p. 31. But it is not a direct product: $D_3 \not\cong C_2 \times C_3$ as the rhs is Abelian, whereas $D_3$ is not!

* Quotients and direct product play an important role in Physics:

- e.g. in the Standard Model of elementary particle physics we have a symmetry group $U(1) \times SU(2) \times SU(3)$
- chiral symmetry breaking in QCD and QCD-like theories

  * for gauge group $SU(N_c \geq 3)$ $\quad \underbrace{SU(N_f)_L \times SU(N_f)_R}_{G} \to \underbrace{SU(N_f)_V}_{H}$

    with the Goldstone bosons $\in$ coset $G/H$

  for $SU(2)$ colours $\quad SU(2N_f) \to SO(2N_f)$

  $\rightsquigarrow$ more complicated coset (cannot divide out $SU(N_f)$ )

# Homomorphisms

We have already called groups equivalent, e.g. $\mathbb{Z}_2 \cong C_2$ and named this an isomorphism. For small finite groups this can be established by comparing their multiplication tables. For larger or infinite groups this becomes impractical, so let's make this more precise.

<u>Def</u> For 2 groups $A$ and $B$ the map $f: A \to B$ is a <u>group homomorphism</u> if it preserves the group multiplication:

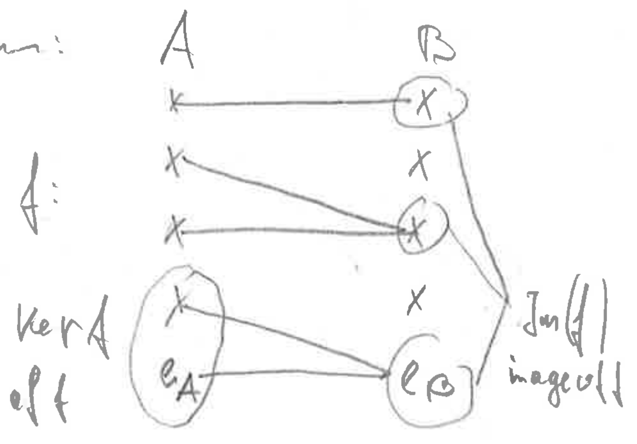
$$\forall a, a' \in A \quad f(\underbrace{a \cdot a'}_{\text{mult in } A}) = \underbrace{f(a) \cdot f(a')}_{\text{mult in } B}$$

If in addition $f$ is bijective it is called a <u>group isomorphism</u> and we write $A \cong B$.

To be a group homomorphism poses some restrictions on $f$, e.g.

- For $e_A, e_B$ identity elements of $A, B$ it holds
  $$\forall a \in A: f(a) = f(a \cdot e_A) = f(a) \cdot f(e_A) \Rightarrow f(e_A) = e_B$$
- $$\forall a, a^{-1} \in A: f(e_A) = f(a \cdot a^{-1}) = f(a) \cdot f(a^{-1}) \Rightarrow f(a^{-1}) = (f(a))^{-1}$$

For $f$ a group homomorphism but not an isomorphism 2 things can happen:




- not all elements of $B$ are the image of a point of $A$ (not surjective)
- <u>different</u> points in $A$ get mapped to the <u>same</u> point in $B$ (not injective)

Both image and kernel of $f$ form a group

The <u>Image Group</u>:

$\mathrm{Im}(f) = \{ b \in B : b = f(a) \text{ for some } a \in A \}$

$\mathrm{Im}(f)$ is a subgroup of $B$: we have already seen that $e_B$, $b^{-1}$ are in $\mathrm{Im} f$. Closure: $b_1, b_2 \in \mathrm{Im}(f) \Rightarrow \exists a_1, a_2 \in A$ s.th. $b_1 = f(a_1)$, $b_2 = f(a_2)$ $\Rightarrow b_1 b_2 = f(a_1) f(a_2) = f(\underset{\curlywedge}{a_1 a_2})$

$\Rightarrow \in \mathrm{Im}(f)$

<u>The kernel group</u>

$\mathrm{Ker} f = \{ a \in A : f(a) = e_B \}$ is a <u>normal</u> <u>subgroup of</u> $A$ $\left( \begin{matrix} e_A \in \ker \\ f(a^{-1}) = f(a)^{-1} = e_B^{-1} \\ = e_B \end{matrix} \right)$

for $a, b \in \ker f$ $\quad f(a \cdot b) = f(a) f(b) = e_B e_B = e_B \Rightarrow a \cdot b \in \ker f$
closure

We need to show $x \ker f = \ker f \, x \quad \forall x \in A$

$\Leftrightarrow \forall a \in \ker f \quad x a x^{-1} \in \ker f : f(x a x^{-1}) = f(x) e_B f(x^{-1})$

$= e_B \qquad$ "$(f(x))^{-1}$"

□

It follows that $A/\ker f$ is a group itself!

This furthermore restricts possible homomorphisms of finite groups. It also follows:

The <u>Isomorphism Theorem</u>: If $f: A \to A'$ is a homomorphism then $\mathrm{Im}(f) \cong A/_{\ker f}$ is an isomorphism