Proof: Define $F: \begin{cases} A/K \to \mathrm{Im}(f) \\ gK \to F(gK) = f(g) \end{cases}$, $K \equiv \ker f$

• F is well defined:

let $gK = g'K \Rightarrow F(gK) = f(g)$, e_B as $a \in K$

now $g = g'a$ with $a \in K \Rightarrow F(gK) = f(g')f(a) = f(g') = F(g'K)$

• F is injective: If $F(gK) = F(g'K) \Rightarrow f(g) = f(g')$

$\Rightarrow f(\underbrace{g'^{-1}g}_{a}) = e \Rightarrow g'^{-1}g \in K \Rightarrow$

$gK = g'\underbrace{g'^{-1}g}_{\in K}K = g'K$

• F is surjective:

we can choose any $g \in A$, consider $F(gK)$ to get any element $f(g)$ in $\mathrm{Im}(f)$

• F is a homomorphism (group mult. preserving)

$$F(gK)\,F(g'K) = f(g)\,f(g') = f(gg') = F(\underline{gg'K})$$

group mult. on gK p.29 $\square$

example: D_3 has only 1 (non-trivial) normal subgroup,

$C_3 = \{e, c, c^2\}$ ($\Leftrightarrow$ a candidate for $\ker f$)

Given a homomorphism $f: D_3 \to G$ for G some group, we can have

- $\ker f = \{e\}$ (is a normal subgroup: the trivial one)

$\Rightarrow \quad \mathrm{Im}(f) \cong D_3/\{e\} = D_3$

- $\ker f = C_3$. We know that $D_3/C_3 = C_2$

$\Rightarrow$ the isomorphism Thm tells us that $\mathrm{Im}(f) \cong C_2$

This illustrates the restrictiveness of the concept of group homomorphisms!

Motivation <u>Representations</u> of Groups:

Consider the hydrogen atom in QM, with Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{r})$. It acts on a Hilbert space with states labelled by quantum numbers n, l, m

$$\hat{H}|n,l,m\rangle \sim \frac{1}{n^2}|n,l,m\rangle, \quad L^2|n,l,m\rangle = l(l+1)|n,l,m\rangle$$
$$L_z|\quad\rangle = m|n,l,m\rangle, \quad m=-l,\dots,+l$$

- a rotation R_1 in 3 dim will change m, but not n or l:

$\Rightarrow$ the rotated state $|n,l,m'\rangle = D^{(l)}_{m'm}(R_1)|n,l,m\rangle$ is a linear superposition, because of the possible values of m. $D(R_1)$ is a matrix of dim $2l+1$

- we do a further rotation R_2 we can write

$$|n,l,m''\rangle = D^{(l)}_{m''m'}(R_2)|n,l,m'\rangle = \underline{D^{(l)}_{m''m'}(R_2)\,D^{(l)}_{m'm}(R_1)}\,|n,l,m\rangle$$

but also directly for the rotation $R = R_2R_1$

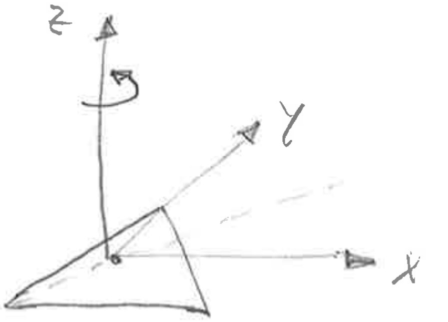
$$= \underline{D^{(l)}_{m'm}(R = R_2R_1)}\,|n,l,m\rangle$$

they form a representation

$\Rightarrow$ the matrices $D(R)$ fill a group mult. table of the underlying group.

Group representations

- the groups we have studied so far were defined on an abstract level by their multiplication table (Latin square): they satisfy the rules G0 – G3 that define a group.
- However, we sometimes had explicit realisations, e.g. for the groups C_3 or D_3 in terms of rotations of a given triangle in $\mathbb{R}^3$.



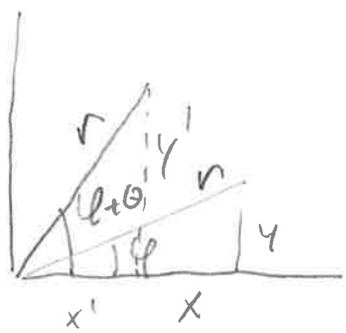
→ these rotations are a typical example of a representation of a group. In detail

- The elements of $C_3 = \{e, c, c^2\}$ can be represented by rotations around the z-axis by an angle of $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$

Such rotations in the xy-plane by an arbitrary angle θ enjoy a 3x3 matrix rep on $\mathbb{R}^3$:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ z \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv R(\theta) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

as



$$x' = r\cos(\varphi+\theta) = r(\cos\varphi\cos\theta - \sin\varphi\sin\theta) \stackrel{!}{=} x\cos\theta - y\sin\theta$$

$$y' = r\sin(\varphi+\theta) = r(\sin\varphi\cos\theta + \cos\varphi\sin\theta) \stackrel{!}{=} y\cos\theta + x\sin\theta$$

using $\sin\left(\frac{2\pi}{3}\right) = \frac{1}{2}\sqrt{3}$, $\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2}\sqrt{3}$ we have
$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$, $\cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$

the following representation of C_3 in terms of 3×3 rotation matrices:

$$D(e) = R(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D(c) = R\left(\tfrac{2\pi}{3}\right) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D(c^2) = R\left(\tfrac{4\pi}{3}\right) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- it is easy to check that indeed

$$D(c^2) = (D(c))^2, \quad D(c)D(c^2) = D(c^3) = D(e) \quad \text{etc.}$$

as in the multiplication table for C_3

- choosing angles $\Theta = \frac{2\pi}{n}k, \quad k = 0, 1, \ldots, n-1$

  we obtain a <u>matrix rep. of C_n</u> by 3×3 rotation matrices
- for D_n we would have matrices rotating around other axes by π (reflections in 2dim)

<u>Definition</u> A <u>matrix representation</u> (short: rep.) of dimension n of the abstract group G is defined as a <u>homomorphism</u> $D: G \to GL(n, \mathbb{C})$.

Here <u>$GL(n, \mathbb{C})$</u>: (general linear group) is the <u>group</u> of non-singular (= invertible) $n\times n$ matrices with complex entries.

(recall that $f: G \to G'$ is a group homomorphism if it preserves the group structure: $\forall g_1, g_2 \in G \quad f(g_1 g_2) = f(g_1) f(g_2)$)

Consequently D preserves the group structure, and it holds

$$D(g^{-1}) = (D(g))^{-1} \qquad (\text{hence } D(g) \text{ must be invertible})$$

In our example C_3 belongs to a more restrictive group:

C_3: $D\left(\frac{2\pi k}{3}\right) \in$ <u>$SL(n, \mathbb{R})$</u> (special linear group) the <u>group</u> of <u>non-singular</u> $n\times n$ matrices with real elements and determinant 1.

$\sqrt{\vec{x}^2} = \|x\|$

- because $R(\theta)$ is a rotation matrix it preserves the norm of any vector, in fact the scalar product of any 2 vectors $\vec{a}\cdot\vec{b}$:

$$\vec{x}'^2 = \vec{x}'^T \vec{x}' = (R\vec{x})^T R\vec{x} = \vec{x}^T R^T R \vec{x} \overset{!}{=} \vec{x}^T \vec{x} \Rightarrow \boxed{R^T R = \mathbb{1}}$$

$$\Rightarrow \det(R^T R) = \det R^T \cdot \det R = (\det R)^2 = 1 \Rightarrow \det R = \pm 1$$

- Because $R(\theta)$ is a proper rotation (and not a reflection) $\underline{\det R(\theta) = +1}$

Also $\underline{R \text{ is orthogonal}}$ as $R^{-1} = R^T$

Hence the $D(\theta)$ representing C_n are a discrete subset of the group of $\underset{(\det = 1)}{\underline{\text{special}}}$ $\underline{\text{orthogonal}}$ 3×3 matrices : $SO(3)$

- $\underline{\text{Def}}$ A representation is called $\underline{\text{faithful}}$ if the group homomorphism $D: G \to GL(n, \mathbb{C})$ is an isomorphism $\Leftrightarrow$ $\ker(D) = e \in G$

$\underline{\text{examples}}$: * the rep. $R(0), R(\frac{2\pi}{3}), R(\frac{4\pi}{3})$ of C_3 is faithful.

* In the case of C_3 we only needed rotations in 2 dimensions.

Hence by dropping the last row and column of $R(\theta)$ we also have a rep $D: C_3 \to SL(2, \mathbb{R})$ in $SO(2)$:

$$R_2(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$ with $R_2(0), R_2\left(\frac{2\pi}{3}\right), R_2\left(\frac{4\pi}{3}\right)$ which is again faithful

* If we were to drop the 1st and 2nd row and coulumn we would have $D: C_3 \to SL(1, \mathbb{R})$ with $D(e) = 1$, $D(c) = 1$, $D(c^2) = 1$, this rep is $\underline{\text{not}}$ faithful

* note that for D_3 we either need rotations $\underline{\text{and}}$ reflections in 2 dim that is $D_3 \to GL(2, \mathbb{R})$ or only — " — but in 3 dim: $D_3 \to SL(3, \mathbb{R})$

Equivalence of representations

<u>Definition</u>: Two representations $D^{(1)}$ and $D^{(2)}$ of the same group, with the same dimension n are called <u>equivalent</u>, if <u>for any</u> $g \in G$ the two matrices $D^{(1)}(g)$ and $D^{(2)}(g)$ are related by the same similarity transformation $S \in GL(n, \mathbb{C})$:

$$\forall g \in G : \quad D^{(1)}(g) = S D^{(2)}(g) S^{-1} \qquad (\text{so } S \text{ is indep of } g)$$

---

* it is easy to check that this similarity trafo indeed provides an <u>equivalence relation</u>

* the similarity trafo S is compatible with the group structure of the two reps: $D^{(i)}(gg') = D^{(i)}(g) D^{(i)}(g')$ for $g, g' \in G$ $(i = 1, 2)$

$$S D^{(2)}(gg') S^{-1} = S D^{(2)}(g) \underset{S^{-1}S = \mathbb{1}}{\wedge} D^{(2)}(g') S^{-1} = D^{(1)}(g) D^{(1)}(g') = D^{(1)}(gg')$$

* later when we classify representations we will count each such equiv. class only once!

<u>Example</u>: consider $T(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$, then $\{T(0), T(\frac{2\pi}{3}), T(\frac{4\pi}{3})\}$ is also a matrix rep. of C_3. It is equivalent to $\{R(0), R(\frac{2\pi}{3}), R(\frac{4\pi}{3})\}$ via $S = ?$

In general equivalent reps will be given by matrices that provide the same linear trafo of the vectorspace $\mathbb{R}^n$, but in different bases.

* the concept of a <u>charcter</u> of a rep. of G provides a way of counting equiv classes just once.

<u>Def</u> The <u>character</u> of a matrix rep D of a group G is the set of $\chi = \{\chi(g) \mid g \in G\}$, where $\chi(g)$ is the trace of the rep. matrix D: $\chi(g) = \mathrm{Tr}(D(g))$

- the cyclicity of the trace, $Tr(ABC) = Tr(CAB) = Tr(BCA)$ immediately implies, that the character of two equivalent reps. is the same:

$$Tr(D^{(2)}(g)) = Tr(S D^{(1)}(g) S^{-1}) = Tr(D^{(1)}(g)) \quad \forall g \in G$$

so $\chi = \chi^1 = \chi^2$, e.g. for $\{R(0), R(\frac{2\pi}{3}), R(\frac{4\pi}{3})\}$ of C_3 we have $\chi = \{3, 0\}$

* A main result in rep. theory is that the reverse is also true, i.e. If $\chi^1 = \chi^2$ for 2 reps of the same dim n, then $D^{(1)}$ and $D^{(2)}$ have to be equivalent.

- in the example of the 3x3 matrix rep. of C_3 we saw, that we could "reduce" it in 2 different ways to a (faithful) 2x2 matrix rep and a 1x1 "matrix" rep. In fact our rep decomposes in these 2 reps which are completely indep. in the following sense:

$$RR' = \begin{pmatrix} A & 0 \\ 0^T & 1 \end{pmatrix} \begin{pmatrix} A' & 0 \\ 0^T & 1 \end{pmatrix} = \begin{pmatrix} AA' & 0 \\ 0^T & 1 \end{pmatrix}$$

where A, A' are 2x2 matrices, and $0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

- We denote such a block diag. decomposition by $\oplus$

$$R(c) = D^{(1)}(c) \oplus D^{(2)}(c)$$

Definition A matrix rep. D of dimension $n+m$ is called reducible, if $\forall g \in G$ $D(g) = \begin{pmatrix} A(g) & C(g) \\ 0 & B(g) \end{pmatrix}$ where A and B are of dim $m \times m$ and $n \times n$ respectively and $C(g)$, 0^T are rectangular matrices of dim $m \times n$, with 0 containing only zeros.

- multiplying upper triangular matrices remains upper triangular

$$D(g)\,D(g') = \begin{pmatrix} A(g) & C(g) \\ 0 & B(g) \end{pmatrix} \begin{pmatrix} A(g') & C(g') \\ 0 & B(g') \end{pmatrix} = \begin{pmatrix} \overbrace{A(g)\,A(g')}^{A(gg')} & \overbrace{A(g)\,C(g') + C(g)\,B(g')}^{C(gg')} \\ 0 & \underbrace{B(g)\,B(g')}_{B(gg')} \end{pmatrix}$$