- the cyclicity of the trace, $\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB) = \mathrm{Tr}(BCA)$ immediately implies, that the character of two equivalent reps. is the same:

$$\mathrm{Tr}\left(D^{(2)}(g)\right) = \mathrm{Tr}\left(S D^{(1)}(g) S^{-1}\right) = \mathrm{Tr}\left(D^{(1)}(g)\right) \quad \forall g \in G$$

so $\chi = \chi^1 = \chi^2$, e.g. for $\{R(0), R(\frac{2\pi}{3}), R(\frac{4\pi}{3})\}$ of C_3 we have $\chi = \{3, 0\}$

* A main result in rep. theory is that the reverse is also true, i.e. if $\chi^1 = \chi^2$ for 2 reps of the same dim n, then $D^{(1)}$ and $D^{(2)}$ have to be equivalent.

- in the example of the 3x3 matrix rep. of C_3 we saw, that we could "reduce" it in 2 different ways to a (faithful) 2x2 matrix rep and a (non faithful) 1x1 "matrix" rep. In fact our rep decomposes in these 2 reps which are completely indep. in the following sense:

$$RR' = \begin{pmatrix} A & 0 \\ 0^T & 1 \end{pmatrix}\begin{pmatrix} A' & 0 \\ 0^T & 1 \end{pmatrix} = \begin{pmatrix} AA' & 0 \\ 0^T & 1 \end{pmatrix}$$

where A, A' are 2x2 matrices, and $0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

We denote such a block diag. decomposition by $\oplus$

$$R(c) = D^{(1)}(c) \oplus D^{(2)}(c)$$

<u>Definition</u> A matrix rep. D of dimension $n+m$ is called <u>reducible</u>, if $\forall g \in G$ $D(g) = \left(\begin{array}{c|c} A(g) & C(g) \\ \hline 0 & B(g) \end{array}\right)$ where A and B are of dim $m \times m$ and $n \times n$ respectively and $C(g)$, 0^T are rectangular matrices of dim $m \times n$, with 0 containing only zeros.

- multiplying block-upper triangular matrices remains upper triangular

$$D(g)\,D(g') = \begin{pmatrix} A(g) & C(g) \\ 0 & B(g) \end{pmatrix}\begin{pmatrix} A(g') & C(g') \\ 0 & B(g') \end{pmatrix} = \begin{pmatrix} \overbrace{A(g)\,A(g')}^{A(gg')} & \overbrace{A(g)\,C(g') + C(g)\,B(g')}^{C(gg')} \\ 0 & \underbrace{B(g)\,B(g')}_{B(gg')} \end{pmatrix}$$

$\Rightarrow$ A and B are matrix reps of dim m and n respectively

* We will show below (Maschke's Thm) that for finite groups C can be taken to be the null matrix as for C_3 above, upon using equivalence.

<u>Def</u> A reducible matrix rep as above is called <u>completely reducible</u> or <u>decomposable</u> if $C \equiv 0$, and we write

$$D(g) = A(g) \oplus B(g)$$

* it can happen that $A(g)$ and $B(g)$ are themselves again decomposable. In that case we can continue this process, until we reach reps. that are no longer composable. Such reps. are called <u>irreducible representations</u> (short: irreps) While there is obviously no limitation to number and dim of reducible reps the irreps can be <u>classified</u> (using characters) and <u>enumerated</u>.

Groups Acting on Vector Spaces:

So far we have encountered matrix reps. acting on vectors in $\mathbb{R}^n$. We defined equivalence classes of reps. acting on the same $\mathbb{R}^n$, and claimed they were related by a change of basis. Hence we will now study more abstract, basis indep. reps., so that we can control only equivalence classes then. Let us begin by recalling some facts from Linear Algebra.

Vector Space Axioms

A *vector space* (over the field $\mathbb{C}$) is a set $\{\vec{v}\}$ with two operations denoted by "+" and "$\cdot$", that satisfies the following axioms.

- V forms an Abelian group w.r.t "+", or in other words

A0) closure : $\forall \vec{u},\vec{v} \in V \quad \vec{u}+\vec{v} \in V$

A1) associativity $\forall \vec{u},\vec{v},\vec{w} \in V \quad \vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$

A2) $\exists$ identity $\vec{0} \in V$: s.th. $\forall \vec{u} \in V \quad \vec{u}+\vec{0} = \vec{u}$

A3) $\exists$ inverse : $\forall \vec{u} \in V$ $\exists$ element $-\vec{u} \in V$ s.th. $\vec{u} + (-\vec{u}) = \vec{0}$

A4) commutativity $\forall \vec{u},\vec{v} \in V \quad \vec{u}+\vec{v} = \vec{v}+\vec{u}$

- V satisfies the following rules wrt multiplication $\cdot$ by scalars $a \in \mathbb{C}$ field:

B0) $\forall a \in \mathbb{C} \; \forall \vec{v} \in V \quad a \cdot \vec{v} \in V$

B1) $\forall a,b \in \mathbb{C} \; \forall \vec{u},\vec{v} \in V$: $\quad a\cdot(\vec{u}+\vec{v}) = a\cdot\vec{u} + a\cdot\vec{v}$

B2) —— " —— $\quad (a+b)\cdot\vec{u} = a\cdot\vec{u} + b\cdot\vec{u}$

B3) $\quad a\cdot(b\cdot\vec{u}) = (ab)\cdot\vec{u}$

B4) $\exists\, 1 \in \mathbb{C}$ s.th. $\forall \vec{v} \in V \quad 1\cdot\vec{v} = \vec{v}$

B1-4 and the fact that $\mathbb{C}$ is a field imply the following

- $(1+\underset{\text{id wrt + in } \mathbb{C}}{0})\cdot\vec{u} = 1\cdot\vec{u} + 0\cdot\vec{u} = \vec{u} \implies \underline{0\cdot\vec{u}} = \vec{u} + (-\vec{u}) = \underline{\vec{0}}$

- $(1+\underset{\text{inv of 1 wrt + in } \mathbb{C}}{(-1)})\cdot\vec{u} = 0\cdot\vec{u} = \vec{0} \implies \underline{(-1)\cdot\vec{u} = -\vec{u}}$ inverse to $\vec{u}$

- A vector space can be finite or infinite dimensional
- A subset $U \subseteq V$ that forms a vector space itself is called a *sub (vector) space*

Linear Independence and Basis

Def A set of vectors $\{\vec{e}_i\}$ $i = 1, \dots, n$ is called linearly indep if $\sum_{i=1}^{n} \lambda_i \vec{e}_i = \vec{0} \Rightarrow \forall i = 1, \dots, n \;\; \lambda_i = 0$. Else it is called linearly dependent.

(so if $\vec{0}$ is in such a set it is always lin. dep.!)

Def A linearly independent set of vectors $\{\vec{e}_i\}_{i=1,\dots,n}$ is called a basis of vectorspace V if it spans V, i.e. any $\vec{u} \in V$ can be written as a linear combination of the elements of $\{\vec{e}_i\}$:

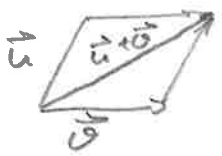
$$\forall \vec{u} \in V \;\; \exists u_1, \dots, u_n \in \mathbb{C} \text{ s.th. } \vec{u} = \sum_{i=1}^{n} u_i \vec{e}_i \equiv \underbrace{u_i \vec{e}_i}_{\text{summation conv.}}$$

Def A vectorspace V is called n-dimensional if it has a basis of n vectors. It is called infinite-dimensional if there is no limit to the number of linearly independent vectors.

Examples: • $\mathbb{R}^3$ with field $\mathbb{R}$ $(\subseteq \mathbb{C})$ is a vector space

— we can define it in a basis indep way:

+ by parallelogram rule , $\vec{0}$ has length 0



and scalar mult. by $a \in \mathbb{R}$: multiplying the length of $\vec{u}$ by a

— in a basis dep. way, say i, j, k for $\{i, j, k\}$ forming a basis

$$\vec{u} = u_1 i + u_2 j + u_3 k \quad \Rightarrow \quad \vec{u}+\vec{v} = (u_1, u_2, u_3) + (v_1, v_2, v_3)$$
$u_j \in \mathbb{R}$
$$= (u_1+v_1,\ u_2+v_2,\ u_3+v_3)$$

and $a \cdot (u_1, u_2, u_3) = (au_1, au_2, au_3)$

$\mathbb{R}^3$ has dimension 3

• an example for an ∞-dim vector space V

V = space of complex valued functions $f: x \in [0,1] \to \mathbb{C}$
on $I = [0,1]$ satisfying $f(0) = f(1) = 0$ & $\int_0^1 dx |f|^2 < \infty$

addition: $(f+g)(x) = f(x) + g(x)$

scalar mult: $(af)(x) = a \cdot f(x)$

This V arises naturally in quantum mechanics for
The energy eigen functions are $u_n(x) = \sin(n x \pi)$, $n \in \mathbb{N}_+$
forming a basis and there is a 1-1 correspondance between
the Fourier modes f_n of f: $f(x) = \sum_{n=1}^{\infty} f_n u_n(x)$

Linear Transformations

Def A map T from the vectorspace onto itself ($V \to V$) is called linear (transform)

if $\forall a,b \in \mathbb{C}, \forall \vec{u},\vec{v} \in V \quad T(a\vec{u}+b\vec{v}) = a\cdot T(\vec{u}) + b\cdot T(\vec{v})$

example: · pick $k \in \mathbb{C}$ and define $T(\vec{u}) = k\cdot\vec{u}$

· on the vectorspace of differentiable (C^∞) functions the derivative is linear

explicit realisations

Given a basis $\{\vec{e}_i\}$ of V linear maps are represented by a matrix

D_{ij} (of dim n if V is of dim n) acting on this basis:

$$\forall j \quad T\vec{e}_j = D_{ij}\vec{e}_i$$

This represents T on all elements $\vec{u} \in V$:

$$T(\vec{u}) = T(u_j\vec{e}_j) = u_j T(\vec{e}_j) = u_j D_{ij}\vec{e}_i = v_i\vec{e}_i = \vec{v} \in V$$

$$\Leftrightarrow \boxed{v_i = D_{ij}u_j} \Leftrightarrow \vec{v} = D\vec{u}$$

Similarity: The matrix D representing T on V is not unique.

Let $\{\vec{e}_i\}_{i=1,\dots,n}$ and $\{\vec{f}_i\}_{i=1,\dots,n}$ be 2 different basis of V.

Then in addition to above we have $T\vec{f}_j = D'_{ij}\vec{f}_i$ and

$$\boxed{v'_i = D'_{ij}u'_j} \Leftrightarrow \vec{v}' = D'\vec{u}'$$

$\Rightarrow$ the two matrices D, D' are related by a similarity trafo S:

express $\{\vec{e}_i\}$ in the new basis $\{\vec{f}_i\}$: $e_i = S_{ji}f_j$ (or $S^{-1}_{ij}\vec{e}_i = \vec{f}_j$)

$\Rightarrow \vec{u} = u_i\vec{e}_i = u_i S_{ji}\vec{f}_j = u'_j\vec{f}_j \Leftrightarrow u'_j = S_{ji}u_i \Leftrightarrow \vec{u}' = S\vec{u}$

$$\Rightarrow \vec{v}' = S\vec{v} = SD\vec{u} = SDS^{-1}\vec{u}' = D'\vec{u}' \Leftrightarrow \boxed{D' = SDS^{-1}}$$

So far we have seen reps of a group as a matrix rep. D, given by the map $D: G \to GL(n, \mathbb{C})$, where D_g is a matrix of the n-dim complex vector space V acting in some basis. We can now lift this rep. to a basis indep. way using the following

Def: Given a vector space V, a group G and a map

$$T: \begin{cases} G \to V \\ g \to T(g) \end{cases}$$ with $T(g)$ being a linear trafo acting on V,

s.th. $\forall g, g' \in G \quad T(g'g) = T(g')\,T(g)$. Such a triple (V, G, T) is called a <u>G-module</u>.

As we have seen lin. maps T can be represented by matrices acting on V in a given basis. Two different reps are related by similarity trafo S: $D'(g) = S\,D(g)\,S^{-1}$. Thus a <u>G-module represents an entire equivalence class</u> of matrix reps.

Def A <u>submodule</u> U is given by a (vector) subspace $U \subseteq V$, where (V, G, T) is a G-module and U is closed under the action of G:

$$\forall g \in G,\ \forall \vec{u} \in U \subseteq V: \quad T(g)\vec{u} \subset U.$$

Given a G-module on V of dim $m+n$, and a submodule U of dim m. Label the bases of V s.th. $\{e_1, \dots, e_m\}$ is a basis of the subspace U, $\{e_i\}_{i=1,\dots,m+n}$ is a basis of V. The lin. trafo T is represented by matrix D on $\{e_i\}$:

$$T(g)\,e_j = D_{ij}(g)\,e_i$$