Symmetries in Physics

I Introduction

• global symmetries: discrete ~

- rotation by a fixed angle α } e.g. in a crystal
- translation " vector $\vec{v}$
- parity (P)
- time reversal (T)
- charge conjugation (C)
- permutations of particles

• global sym.: continuous

- compact (depends on parameters $\in$ a set):
  * rotations ($SO(3)$), internal sym eg. spin ($SU(2)$)
    $\rightarrow$ group theory (part II of this course)
- non-compact (depend on parameter eg. in $\mathbb{R}$)
  * translations in time, in space } Poincaré trafo } part III
  * Lorentz trafo (boost)
  * conformal symmetry (is angle preserving)

• local sym = gauge sym: different at each point in space-time

e.g. multiplication by phase $e^{i\varphi} \rightarrow e^{i\varphi(x)}$ now x-dep

• graded symmetry (Super sym.): $\exists$ global & local versions

! the set of all possible sym. of a "reasonable" QFT is restricted

"No-Go Thm": only Poincare $\otimes$ internal sym $\rightarrow$ SUSY
(+ conf)

Symmetry breaking – examples

spontaneous SB:

a) magnetisation in the ground state of a ferro-magnet $H \sim -J \sum \vec{S} \cdot \vec{S}$

b) chiral symmetry in QCD

$\Psi = \binom{u}{d}$, $\Psi_{R,L} = \frac{1}{2}(\mathbb{1} \pm \gamma_5)\Psi$

$\bar{\Psi} \not{\partial} \Psi = \bar{\Psi}_R \not{\partial} \Psi_R + \bar{\Psi}_L \not{\partial} \Psi_L$

has $\Psi_{R,L} \to U_{R,L} \Psi_{R,L}$, $U_{R,L} \in U(2)$

broken by chiral condensate

$\Sigma = \langle \bar{\Psi}\Psi \rangle = \langle \bar{\Psi}_R \Psi_L + \bar{\Psi}_L \Psi_R \rangle \neq 0$

c) more general: Goldstone- (global) and Higgs-mechanism (local)

(Brout † Englert + Higgs Nobelprice 2013 (history → web)

→ influences the spectrum of massive/massless particles in our theory at hand

explicit SB:

apply external magnetic field $\vec{B} \neq \vec{0}$

addition of an explicit mass term

$m\bar{\Psi}\Psi = m(\bar{\Psi}_R \Psi_L + \bar{\Psi}_L \Psi_R)$

$\to \quad U_R = U_L$

"$U(2) \times U(2) / U(2)$"

anomalies: breaking of a symmetry of the classical theory (Lagrangian) by quantum effects

gauge fixing in a gauge theory we may have too many d.o.f. (due to sym.), e.g. photon has 2 (not 4) → add gauge fixing term to the Lagrangian

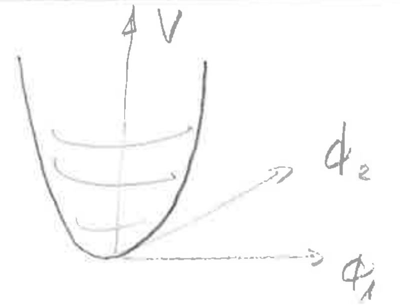
Goldstone-mechanism: breaking of a global symmetry

example: 2 real scalar fields $\phi_{1,2} \Leftrightarrow \phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}$

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) \underbrace{- m^2 \phi\phi^* - \lambda(\phi\phi^*)^2}_{-V(\phi\phi^*)} \qquad \phi^4\text{-theory} \qquad \boxed{\text{2 massive fields}}$$

• usually $\langle\phi_1\rangle = \langle\phi_2\rangle = 0$ (sym. phase).

minimum at $\phi = 0$, $m^2, \lambda > 0$



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• breaking for $m^2 < 0$: minimum at $\phi\phi^* = -\frac{m^2}{2\lambda} \equiv a^2$

choose $\langle\phi\rangle = a$ so $\langle\phi_1\rangle \neq 0$ vev $\langle\phi_2\rangle = 0$

define new fields $\boxed{\phi(x) = a + \frac{\phi_1'(x) + i\phi_2'(x)}{\sqrt{2}}}$, $\langle\phi_{1,2}'\rangle = 0$

$$\Rightarrow \mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1)\partial^\mu\phi_1 + \frac{1}{2}(\partial_\mu\phi_2)(\partial^\mu\phi_2) + \lambda a^4$$

$$\underbrace{-2a^2\lambda\phi_1^2}_{\text{standard mass term}} \quad \underbrace{-\sqrt{2}\lambda a\phi_1(\phi_1^2+\phi_2^2) - \frac{\lambda}{4}(\phi_1^2+\phi_2^2)^2}_{\text{interactions}}$$

ϕ_1 1 massive scalar field

ϕ_2 1 massless " " : Goldstone Boson (GB)

Here: broken U(1) symmetry = multiplication with phase (global)
$\phi(x) \to e^{i\varphi}\phi(x)$

In general: need concept of "quotient group" G/H
Sym. before/after breaking to count # of GB

Higgs-mechanism: breaking of a local sym.

repeat for 2 massive real scalars + 1 Abelian gauge field $A^\mu(x)$, massless

$$\mathcal{L} = (D_\mu\phi)(D^\mu\phi)^* - V(\phi\phi^*) - \frac{1}{4} F^{\mu\nu}F_{\mu\nu}$$

where $D_\mu = \partial_\mu - iA_\mu$ $\Rightarrow$ $\exists$ local sym. $\phi(x) \to e^{i\Lambda(x)}\phi(x)$

see ex 1.3

- repeat sym. breaking

$$\phi \to a + \frac{\phi_1' + i\phi_2'}{\sqrt{2}}$$

$$\Rightarrow \mathcal{L} \to \mathcal{L} + a^2 A_\mu A^\mu + \sqrt{2}\, a A^\mu \partial_\mu \phi_2 + \text{interactions}$$

new mass term for A_μ (previously forbidden) decay of photon ?

remove term by infinitesimal local (gauge) trafo

$$\phi_1'' = \phi_1' - \Lambda\phi_2'$$
$$\phi_2'' = \phi_2' + \Lambda\phi_1' + \sqrt{2}\,\Lambda a$$

choose $\Lambda(x)$ to make $\phi_2'(x) = 0$

$\Rightarrow$ resulting theory has 1 massive gauge field
& 1 " scalar " left

$\exists$ more general setups, e.g. in standard model,
ϕ giving masses to Fermions and gauge-fields ($W_\pm$, Z) !

(more reading e.g. QFT: Lewis H Ryder, chapter 8)

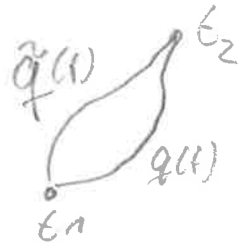
Lagrange-Formalism and conserved charges

1 dimension – point particle $q(t)$, t time

Lagrangian $L(q(t), \dot q(t), t)$ e.g. $L(q,\dot q) = \frac{m}{2}\dot q(t)^2 - V(q(t))$

q ↑ coord. $\dot q$ ↑ velocity — in general no explicit t-dependence

action $S[q] = \int_{t_1}^{t_2} dt\, L(q(t), \dot q(t))$



equations of motion (eom)

vary coord. $q(t) \to \tilde q(t) = q(t) + \delta q(t)$

with fixed end points $\delta q(t_1) = 0 = \delta q(t_2)$ such that (s.th.)

the action is minimized:

$$\frac{\delta S[q]}{\delta q(t)} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial L}{\partial q(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot q(t)} = 0}$$ Euler-Lagrange eqs.

Hamiltonian $H(p,q) = p\dot q - L(q, \dot q(p,q))$

with canonically conjugate momentum $p = \frac{\partial L}{\partial \dot q}$

example above: $p = m\dot q \Rightarrow H = \frac{p^2}{2m} + V(q)$ total energy

action $S = \int_{t_1}^{t_2} dt\, (p\dot q - H(p,q)) = \underset{\substack{\parallel \\ p_2}}{p(t_2)}\,\underset{\substack{\parallel \\ q_2}}{q(t_2)} - p(t_1)q(t_1) - \int_{t_1}^{t_2} dt\, (q(t)\dot p(t) + H)$ etc.

this implies conserved charges if we require invariance under

- time translation $t \to t + \tau$ $\Leftrightarrow$ energy conservation
- space " $q \to q + a$ $\Leftrightarrow$ momentum "

(in >1D: rotations $\vec q$ $\Leftrightarrow$ angular momentum ") consider: $\delta S = S(q_2+\delta q_2, t_2+\delta t_2; q_1+\delta q_1, t_1+\delta t_1) - S(q_2, t_2; q_1, t_1)$

$$= (p_2+\delta p_2)(q_2+\delta q_2) - (p_1+\delta p_1)(q_1+\delta q_1) - \int_{t_1+\delta t_1}^{t_2+\delta t_2} dt\left(q(t)\frac{dp}{dt} + H\right)dt$$
$$- p_2 q_2 + p_1 q_1 + \int_{t_1}^{t_2} dt\,(q\dot{p} + H)$$
$$= p_2\delta q_2 + \delta p_2 q_2 - p_1\delta q_1 - \delta p_1 q_1 - \delta t_2\, q_2\frac{dp_2}{dt} + \delta t_1\, q_1\frac{dp_1}{dt} - \delta t_2 H_2 + \delta t_1 H_1 + O(\delta^2)$$

$$\Leftrightarrow \boxed{\delta S \approx p_2\delta q_2 - p_1\delta q_1 - \delta t_2 H_2 + \delta t_1 H_1} \quad \text{for small } \delta$$

- time: $t_{1,2} \to t_{1,2} + \tau$: $\delta t_{1,2} = \tau$, $\delta q_{1,2} = 0$ $\Rightarrow \delta S \overset{!}{=} 0 \Leftrightarrow$ $\underline{H_2 = H_1}$ energy conservation

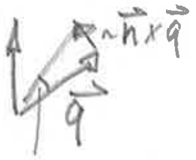
- space: $q_{1,2} \to q_{1,2} + a$: $\delta q_{1,2} = a$, $\delta t_{1,2} = 0$ $\Rightarrow \delta S \overset{!}{=} 0 \Leftrightarrow$ $\underline{p_2 = p_1}$ momentum cons.

correspondingly in dim ≥ 1: $\vec{q}_{1,2} \to \vec{q}_{1,2} + \vec{a}$ in arbitrary directions $\Rightarrow \vec{p}_2 = \vec{p}_1$

- rotations:

$$\vec{q}_{1,2} \to \vec{q}_{1,2} + (\delta\alpha)\,\vec{n}\times\vec{q} = \vec{q}_{1,2} + \delta\vec{q}_{1,2}, \quad \delta t_{1,2} = 0$$

around $\vec{n}$



by angle $\delta\alpha$

$$q(t_{1,2})_i \to \Big(\delta_{ik} + \underbrace{\delta\alpha\,\varepsilon_{ijk} n_j}_{\omega_{ik} = -\omega_{ki}}\Big)\, q(t_{1,2})_k$$

$$\delta S = 0 \Rightarrow \delta\alpha\,\vec{p}(t_2)\cdot(\vec{n}\times\vec{q}(t_2)) = \delta\alpha\,\vec{p}(t_1)\cdot(\vec{n}\times\vec{q}(t_1))$$

cyclicity of product $\Leftrightarrow$ $\vec{n}\cdot(\vec{q}(t_2)\times\vec{p}(t_2)) = \vec{n}\cdot(\vec{q}(t_1)\times\vec{p}(t_1))$

for arb. rotation axis $\vec{n} \Rightarrow$ $\underline{\vec{q}(t_2)\times\vec{p}(t_2) = \vec{q}(t_1)\times\vec{p}(t_1)}$ $\sim$ angular momentum conserv.

example invariant action in arb. dim $\vec{q}\in\mathbb{R}^d$, $|\vec{q}|$ Euclid norm
$$\boxed{L[\vec{q}, \dot{\vec{q}}] = \frac{m}{2}\dot{\vec{q}}\cdot\dot{\vec{q}} - V(|\vec{q}|)}$$