So far we have seen reps of a group as a matrix rep. D, given by the map $D: G \to GL(n, \mathbb{C})$, where D_g is a matrix of the n-dim complex vectorspace V acting in some basis. We can now lift this rep. to a basis indep. way using the following

<u>Def</u>: Given a vectorspace V, a group G and a map

$$T: \begin{cases} G \to Gl(V) \\ g \to T(g) \end{cases}$$

with $T(g)$ being a linear trafo acting on V,

s.th. $\forall g, g' \in G \quad T(g'g) = T(g')\,T(g)$. Such a triple (V, G, T) is called a <u>G-module</u>.

As we have seen lin. maps T can be represented by matrices acting on V in a given basis. Two different reps are related by similarity trafo S: $D'(g) = S\,D(g)\,S^{-1}$. Thus a <u>G-module represents an entire equivalence class</u> of matrix reps.

<u>Def</u> A <u>submodule</u> U is given by a (vector) subspace $U \subseteq V$, where (V, G, T) is a G-module and U is closed under the action of G:

$$\forall g \in G,\ \forall \vec{u} \in U \subseteq V: \quad T(g)\vec{u} \in U.$$

Given a G-module on V of dim $m+n$, and a submodule U of dim m. Label the basis of V s.th. $\{e_1, \dots, e_m\}$ is a basis of the subspace U, $\{e_i\}_{i=1,\dots,m+n}$ is a basis of V. The lin. trafo T is represented by matrix D on $\{e_i\}$:

$$T(g)\,e_j = D_{ij}(g)\,e_i$$

The fact that U is a submodule implies that

$$\begin{array}{c|cc} & m & n \\ \hline m & & \\ n & & \end{array} \quad D(g) = \begin{pmatrix} A(g) & C(g) \\ 0 & B(g) \end{pmatrix} \quad \text{or } D_{ij}(g) = 0 \text{ for } \begin{matrix} i = m+1, \dots, n \\ j = 1, \dots, m \end{matrix}$$

so that elements in $\{e_{i=1..m}\} \subseteq U$ cannot be mapped onto basis elements that are not in the basis of U.

→ This is exactly the case we had for reducible matrix reps!

* In order to show that $C(g) \equiv 0$ can be chosen $\forall$ finite groups we need to show that all their reps. are unitary. To do so we endow V with a (complex)

Def Scalar product. The map $\begin{cases} V \times V \to \mathbb{C} \\ u, v \in V \to (u, v) \end{cases}$ is called so if it has the following properties

complex conj.

S1) hermiticity $\quad \forall \vec{u}, \vec{v} \in V \quad (\vec{u}, \vec{v}) = (\vec{v}, \vec{u})^*$

S2) linearity $\quad \forall \alpha, \beta \in \mathbb{C}, \forall \vec{u}, \vec{v}, \vec{w} \in V \quad (\vec{w}, \alpha\vec{u} + \beta\vec{v}) = \alpha(\vec{w}, \vec{u}) + \beta(\vec{w}, \vec{v})$

S3) positivity $\quad \forall \vec{u} \in V: (\vec{u}, \vec{u}) \geq 0$ with $= 0$ iff $\vec{u} = \vec{0}$.

* in our examples we have: $u_i v_i = (\vec{u}, \vec{v})$ in $\mathbb{R}^3$, with $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$ is the standard scalar product.

* for the ∞-dim space of functions on $[0,1]$ we have

$$(\psi, \varphi) = \int \psi^*(x) \varphi(x) \, dx \text{ as a scalar product}$$

Def The norm of a vector $\vec{u} \in V$ is given by $\|\vec{u}\| \equiv (u, u)^{\frac{1}{2}}$, $\vec{u}$ is called normalised if $\|\vec{u}\| = 1$, and $\vec{u}, \vec{v} \in V \setminus \{\vec{0}\}$ are called orthogonal if $(\vec{u}, \vec{v}) = 0$

Given any basis $\{f_i\}$ of V it is possible to construct an orthonormal basis $\{e_i\}$ of V s.th. $(e_i, e_j) = \delta_{ij}$ (ON basis) (proof: Gram-Schmidt).

In such an ON basis it holds:

- the coord. u_i of $\vec{u} \in V$ are given by $u_i = (e_i, \vec{u})$
- the scalar product of $\vec{u}, \vec{v} \in V$ is given by $(\vec{u}, \vec{v}) = u_i^* v_i$
- a linear map is represented by $Te_j = D_{ij} e_i$, $D_{ij} = (e_i, Te_j)$

and $\forall \vec{u}, \vec{v} \in V$ $(\vec{u}, T\vec{v}) = u_i^* D_{ij} v_j$

Def A linear trafo T on V is called unitary if

$$\forall \vec{u}, \vec{v} \in V \quad (T\vec{u}, T\vec{v}) = (\vec{u}, \vec{v}) \quad \text{, or} \quad (T\vec{u}, \vec{v}) = (u, T^{-1}\vec{v})$$

On the level of the corresp. matrix rep. D in the ON basis $\{e_i\}$ we have

$$(e_i, D^{-1} e_j) = (D^{-1})_{ij} = (De_i, e_j) = (e_j, De_i)^* = D_{ji}^* = (D^T)^*_{ij}$$

or $D^{-1} = D^{T*} = D^{\dagger}$ which defines a unitary matrix.

Def A linear Trafo T on V is called Hermitian if

$$\forall \vec{u}, \vec{v} \in V \quad (T\vec{u}, \vec{v}) = (\vec{u}, T\vec{v})$$

Repeating the above in the ON basis $\{e_i\}$ we obtain that the corresp. matrix D satisfies $D^{\dagger} = D^{T*} = D$.

We can now establish reducibility in a coordinate free language. A reducible rep. was equivalent to having a submodule U of module V that is closed under group action.

Given an ON basis $\{e_i\}_{i=1,\dots,m}$ of U an extension of this to an ON basis $\{e_i\}_{i=1\dots m+n}$ of V where $\{e_{i=m+1\dots n}\}$ spans the complement $W = V \setminus U$, we have

$$W = \{\vec{w} \in V \mid (\vec{w}, \vec{u}) = 0 \;\; \forall \vec{u} \in U\}$$

Now complete reducibility, that is $C(g) = 0$ in $D = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is equivalent to saying that both W and U are closed under the action of G. This is always the case if $T(g)$ is unitary w.r.t. the scalar product we use:

$$\Rightarrow (\underbrace{T(g)\vec{w}}_{\equiv \vec{w}'}, \vec{u}) = (\vec{w}, T^{-1}(g)\vec{u}) = (\vec{w}, \vec{u}') = 0$$

, closure of U: $T^{-1}(g)\vec{u} = T(g^{-1})\vec{u} = \vec{u}' \in U$

$\Rightarrow \vec{w}' \in W$ so W is closed under T too. This implies that $C(g) = 0$.

Maschke's Theorem: All reducible reps of a finite group are completely reducible (= decomposable).

Proof: we have just shown, that reducible reps that are unitary are decomposable. Hence we need to construct a scalar product s.t. the linear map T is represented unitarily:

Define $\{\vec{v}, \vec{v}'\} = \frac{1}{[g]} \sum_{g \in G} (T(g)\vec{v}, T(g)\vec{v}')$ for any $\vec{v}, \vec{v}' \in V$

for G finite group, of order $[g]$. Consider $h \in G$, $\vec{v}, \vec{v}' \in V$ arbitrary

$$\Rightarrow \{T(h)\vec{v}, T(h)\vec{v}'\} = \frac{1}{[g]} \sum_{g \in G} (\underbrace{T(h)T(g)}_{T(hg)}\vec{v}, \underbrace{T(h)T(g)}_{T(hg)}\vec{v}') = \frac{1}{[g]} \sum_{g' \in G} (T(g')\vec{v}, T(g')\vec{v}') = \{\vec{v}, \vec{v}'\}$$

recall that hg generates a permutation of all group elements when g runs through G.

□

In the proof it was important that the sum over all $g \in G$ exists, which is ensured here by the finiteness of G. For compact groups to be def later this is also true.

Df: The scalar product we introduced in the proof above is called group-invariant scalar product.

- Rephrasing Maschke's Thm in form of matrix reps, a reducible rep $T(h)$ will be represented by a reducible unitary matrix $D'(h)$, (that is hence completely reducible) with a corresponding ON basis $\{f_i\}_{i=1..m+n}$ where $\{f_{i=1,..,n}\}$ span the submodule U. $D'(h)$ relates to an equiv. class of other (reducible) matrix rep. $D(h)$ through a similarity trafo S, $D'(h) = S D(h) S^{-1}$, with basis $\{e_i\}_{i=1,..,m+n}$. Thus any reducible matrix rep. of G finite is equivalent to a completely reducible rep.

Properties of Irreducible Representations

- We will now focus on the building blocks of reducible reps, the irreducible reps. These have remarkable properties, the first of which are summarised by 2 Lemmas due to Schur:

1. Lemma:

(Matrix rep formulation): Given an irred. matrix rep D of G in $GL(n, \mathbb{C})$ and an $n \times n$ matrix B. It then holds:

$$\text{If } \forall g \in G : \quad B D(g) = D(g) B \quad \Rightarrow \quad B = \lambda \mathbb{1}_{n \times n}$$

(λ can be zero)