In the proof it was important that the sum over all $g \in G$ exists, which is ensured here by the finiteness of G. For compact groups to be def later this is also true.

Df: The scalar product we introduced in the proof above is called <u>group-invariant scalar product</u>.

- Rephrasing Maschke's Thm in form of matrix reps, a reducible rep $T(h)$ will be represented by a reducible unitary matrix $D'(h)$ (that is hence completely reducible) with a corresponding ON basis $\{f_i\}_{i=1..m+n}$ where $\{f_{i=1..n}\}$ span the submodule U. $D'(h)$ relates to an equiv. class of other (reducible) matrix rep. $D(h)$ through a similarity trafo S, $D'(h) = S D(h) S^{-1}$, with basis $\{e_i\}_{i=1,..,m+n}$. Thus any reducible matrix rep. of G is equivalent to a completely reducible rep.

Properties of Irreducible Representations

- We will now focus on the building blocks of reducible reps, the irreducible reps. These have remarkable properties, the first of which are summarised by 2 Lemmas due to Schur: [Issai Schur, 1875-1941]

<u>1. Lemma:</u>

(Matrix rep formulation): Given an irred. matrix rep D of G in $GL(n, \mathbb{C})$ and an $n \times n$ matrix B. If then holds:

$$\text{If } \forall g \in G : \quad B D(g) = D(g) B \quad \Rightarrow \quad B = \lambda \mathbb{1}_{n \times n}$$

(λ can be zero)

(linear map formulation): Given an irred. G-module with G. map T, vect. space V, and another linear map $\hat{B}: V \to V$ ($\neq$ null map), then we have

$$\forall g \in G: \quad \hat{B}\, T(g) = T(g)\, \hat{B} \quad \Rightarrow \quad \hat{B} = \lambda E \qquad \text{with } E: \vec{u} \to \vec{u} \text{ the identity}$$

composition of the 2 linear maps on V

Proof: Every linear map $\hat{B}$ has at least 1 eigenvector: to find an eigenvalue & eigenvector we have to solve $\det(B - \lambda \mathbb{1}) = 0$ for the matrix rep B. Due to the Fundamental Theorem of Algebra this polynomial has at least 1 solution $\lambda \in \mathbb{C}$. Let $\vec{b} \neq \vec{0}$ be its eigenvect.:

$$\hat{B}\vec{b} = \lambda \vec{b} \quad \Rightarrow \quad \hat{B}\left(T(g)\, \vec{b}\right) \overset{B,T \text{ commute}}{=} T(g)\, \hat{B}\, \vec{b} = T(g)(\lambda \vec{b}) = \lambda \left(T(g)\vec{b}\right)$$

$\Rightarrow$ $T(g)\vec{b}$ is also an eigenvector of $\hat{B}$, with the same eigenvalue λ.

- the space of eigenvectors of $\hat{B}$ is a vector space itself, let's call it U, and thus a subvectorspace of V. According to the above is U closed under the action of the group G via T, so U is a G-module.

Because the G-module (V, G, T) was irreducible we can only have $U = V$ or $U = \{\vec{0}\}$. The latter cannot happen as we had at least 1 eigenvector $\vec{b} \neq \vec{0}$, its span is not the null vector.

$\Rightarrow$ the space of eigenvectors of $\hat{B}$ spans all of V, all with ev λ:

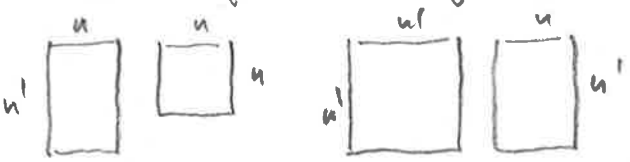
$$\forall \vec{u} \in V: \quad \hat{B}\vec{u} = \hat{B}\, u_i \vec{e}_i = u_i \hat{B} \vec{e}_i = u_i \lambda \vec{e}_i = \lambda \vec{u}$$

$\Rightarrow$ $\hat{B} = \lambda E$ proportional to the identity.

2. Lemma

(matrix form) Consider two <u>inequivalent</u> irreducible matrix reps D and D' of dimensions n and n' respectively, and a matrix B of size $n' \times n$. It holds: if

$$\forall g \in G \quad B\, D(g) = D'(g)\, B \quad \Rightarrow \quad B = 0_{n' \times n}$$



(linear op. form) Consider two inequiv irreducible G modules (U, G, T) and (U', G, T') with U, U' of dim n and n' respectively. Let $\hat{B}$ be a lin operator $\hat{B} : U \to U'$. If

$$\forall g \in G \quad \hat{B}\, T(g) = T'(g)\, \hat{B} \quad \Rightarrow \quad \hat{B} = \hat{0}$$

the null-map ~~that~~ that maps all $\vec{u} \in U$ to zero
$\hat{0}\, \vec{u} = \vec{0}' \quad \forall \vec{u} \in U$.

For the proof distinguish 3 cases:

1) <u>$n < n'$:</u>

Consider $T'(g)\, \hat{B}$ acting on an arbitrary vector $\vec{u} \in U$:

$\forall g \in G$: $T'(g)(\hat{B}\, \vec{u}) = \hat{B}\, \underbrace{T(g)\, \vec{u}}_{\in U}$, so the image of $\hat{B}$ acting on U is <u>closed</u> under the group action of $T'(g)\ \forall g$.

$\Rightarrow$ $\hat{B}\, U$ is a submodule of U'. Because U' is irreducible we have that either a) $\hat{B}\, U = U'$, or b) $\hat{B}\, U = \{\vec{0}'\}$. We can exclude a) because the dim m of the image of $\hat{B}$ of U cannot exceed that of n: $m \leq n < n'$.

$\Rightarrow$ $\hat{B}$ is the null-map as claimed.

2) <u>$n > n'$:</u>

Let us focus on the <u>kernel K</u> of the map $\hat{B}$:

$$K = \{\vec{k} \in U \mid \hat{B}\, \vec{k} = \vec{0}'\}$$

$\mathcal{K}$ is a submodule of $\mathcal{U}$ as it is invariant under the group action of $T'(g)$:

$\forall g \in G \ \forall \vec{k} \in \mathcal{K}$: $\hat{B}(T(g)\vec{k}) = T'(g)\,\hat{B}\vec{k} = \vec{0}'$

Now $\mathcal{U}$ is irreducible, so either a) $\mathcal{K} = \mathcal{U}$ or b) $\mathcal{K} = \{\vec{0}\}$.

Case b) cannot be true: $\hat{B}$ reduces the dimension from n to n', so that all images of the basis $\{\vec{e}_i\}$ of $\mathcal{U}$ can be linearly indep. after acting with $\hat{B}$ $e_{i=1,\dots,n}$ on them as $n > n'$. Hence $\mathcal{K}$ is non trivial and so $\mathcal{K} = \mathcal{U}$, meaning that all $\vec{u} \in \mathcal{U}$ are mapped to $\vec{0}'$ by $\hat{B}$.

2) $n = n'$:

The kernel $\mathcal{K}$ is again a submodule of $\mathcal{U}$, leaving possibilities a) and b) above. Here b) is excluded due to the inequivalence of D and D': Suppose b) is true $\Rightarrow$ $\hat{B}\vec{u} = \hat{B}\vec{v} \Leftrightarrow \vec{u} - \vec{v} \in \mathcal{K}$ so $= \vec{0}$, hence $\hat{B}$ is 1-1 and can be inverted. Consequently we would have a matrix rep. $\forall g \in G$: $B\,D(g)\,B^{-1} = D'(g)$ which is not allowed
$\Rightarrow$ a) is true and $\hat{B}$ is the null-map.

Schur's lemmas allow to state the following

Fundamental Orthogonality Theorem :

Let $\mathcal{U}_\nu, \mathcal{U}_\mu$ be two G-modules $(\mathcal{U}_\nu, G, T^{(\nu)}), (\mathcal{U}_\mu, G, T^{(\mu)})$ carrying inequiv. irreps., with ν, μ labelled by some positive integer. Take any linear map $\hat{A}: \mathcal{U}_\nu \to \mathcal{U}_\mu$

For finite (and compact) groups the following operator is well defined:

$$\hat{B} \equiv \sum_{g \in G} T^{(\mu)}(g)\,\hat{A}\,T^{(\nu)}(g^{-1})$$

It holds that $T^{(\mu)}(h)\,\hat{B} = \hat{B}\,T^{(\nu)}(h) \quad \forall h \in G$ for the $\hat{B}$ we constructed

$$T^{(\mu)}(h)\hat{B} = \sum_{g\in G} \underbrace{T^{(\mu)}(h)\, T^{(\mu)}(g)}_{\substack{T^{(\mu)}(hg) \\ \hat{=}\, g'\in G,\ g'^{-1} = g^{-1}h^{-1}}} \hat{A}\, T^{(\nu)}(g^{-1}) = \sum_{g'\in G} T^{(\mu)}(g')\hat{A}\underbrace{T^{(\nu)}(g'^{-1}h)}_{\hat{=}\, T^{(\nu)}(g'^{-1})T^{(\nu)}(h)}$$

$$= \hat{B}\, T^{(\nu)}(h)$$

2.La

$\Rightarrow \quad \hat{B} = \hat{O} \quad$ for $\mu \neq \nu$

- in case we have $\mu = \nu$ the 1.La tells us that $\hat{B} = \lambda E$, so in matrix form

$$\boxed{B \equiv \sum_{g\in G} D^{(\mu)}(g)\, A\, D^{(\nu)}(g^{-1}) = \lambda_A^{(\mu)} \delta^{\mu\nu}\, \mathbb{1}}$$

where the eigenvalue λ depends on matrix A and rep μ

- So far A was not specified, now choose $A_{rs} = 1$ and all other matrix elements to be zero: $A_{em} = \delta_{er}\delta_{ms}$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\Rightarrow B_{ij} = \sum_{g\in G} D^{(\mu)}_{ie}(g)\, A_{em}\, D^{(\nu)}_{mj}(g^{-1}) = \sum_{g\in G} D^{(\mu)}_{ir}(g)\, D^{(\nu)}_{sj}(g^{-1}) = \lambda^{(\mu)}_{rs}\delta^{\mu\nu}\delta_{ij}$$

* we can now determine λ by taking $\mu = \nu$ and the trace of ij: $\sum_i$ with $i=j$

$$\Rightarrow \sum_{g\in G} \big(\underbrace{D^{(\mu)}(g^{-1})\, D^{(\mu)}(g)}\big)_{sr} = \lambda^{(\mu)}_{rs} \cdot n_\mu,$$ where n_μ is the dim of $D^{(\mu)}$ (from δ_{ii})

also $D^{(\mu)-1}(g)\, D^{(\mu)}(g) = \mathbb{1} \Rightarrow \sum_{g\in G} \delta_{sr} = \lambda^{(\mu)}_{rs} n_\mu$

$$= [g]\,\delta_{rs} = \lambda^{(\mu)}_{rs} n_\mu \Leftrightarrow \lambda^{(\mu)}_{rs} = \frac{\delta_{rs}[g]}{n_\mu}$$

as a final result we have

$$\boxed{\sum_{g\in G} D^{(\mu)}_{ir}(g)\, D^{(\nu)}_{sj}(g^{-1}) = \frac{[g]}{n_\mu}\delta^{\mu\nu}\delta_{rs}\delta_{ij}}$$

- We have shown previously in the proof of Maschke's Thm that Completely reducible (decomposable) representations can be chosen to be unitary, upon using an equivalence relation.

$\Rightarrow$ without loss of generality we may choose a unitary irrep $D^{(\mu)}$ in the previous result: using $D^{(\nu)}(g^{-1})_{ij} = D^{(\nu)}(g)^{-1}_{ij} = D^{(\mu)}(g)^{*}_{ji}$ we can write

$$\sum_{g\in G} D^{(\mu)}_{ir}(g)\, D^{(\nu)}_{js}(g)^{*} = \frac{[g]}{n_\mu}\, \delta^{\mu\nu}\delta_{ij}\delta_{rs}$$

- first let us look at $\mu=\nu$: the sum over g runs from g_1, g_2 to $g_{[g]}$ and for fixed i and r we can view $\left(D^{(\mu)}_{ir}(g_1), \ldots, D^{(\mu)}_{ir}(g_{[g]})\right)$ as a $[g]$-dim column-vector. The LHS can thus be seen as a scalar product between two such vectors, and they are all orthogonal for different index pairs ir. Since $i, r = 1, \ldots, n_\mu$ there are n_μ^2 such orthogonal vectors.

- now take $\mu\neq\nu$:

the LHS is still a scalar product as the lenght of the vectors $[g]$ hasn't changed. Only the number of such vectors has, for $D^{(\mu)}$ it's n_μ^2 and for $D^{(\nu)}$ it's n_ν^2. The vectors made from $D^{(\nu)}$ are still orthogonal wrt those made of $D^{(\mu)}$, and there can be at most n_μ^2 linearly indep. of such vectors for each μ. Summing over all possible irreps μ we get at most $\sum_\mu n_\mu^2$ such indep vectors. However, this number cannot possibly exceed the dimension $[g]$ of the vectors: $\boxed{\sum_\mu n_\mu^2 \leq [g].}$

Because any irrep is at least of dimension $1 \leq n_\mu$ for all μ, we have the inequality $\boxed{\sum_\mu \leq \sum_\mu n_\mu^2 \leq [g]}$

$\Rightarrow$ The number of irreps is finite for finite groups G and it is bounded by the dimension $[g]$ of the group G.
(we will show later that $\sum_\mu n_\mu^2 = [g]$).

Orthogonality of characters

- We had introduced the character of a matrix rep. D by the set of all its traces $\chi = \{\chi(g) = \operatorname{tr} D(g) \mid g \in G\}$. These have the following properties

(i) χ is the same for equivalent matrix reps D, D':
$D'(g) = S\, D(g)\, S^{-1}$ $\forall g \in G$ (as we had already seen due to the cyclicity of the trace)

(ii) χ is the same for conjugate group elements:
$g, h \in G$: $D(hgh^{-1}) = D(h)\, D(g)\, D(h)^{-1}$, again use that trace is cyclic

(iii) if D is unitary, i.e. $D^{-1} = D^{+}$, then

$$\chi(g^{-1}) = \operatorname{Tr}\left(D(g)^{-1}\right) = \operatorname{Tr}\left(D(g)^{+}\right) = \chi(g)^*$$

The latter is thus always true for completely reducible reps as we may choose an equiv. unitary rep. Taking the trace over both matrices in the relation on p. 50 we thus have

$$\sum_{g \in G} \chi^{(\mu)}(g)\, \chi^{(\nu)}(g^{-1}) = \frac{[g]}{n_\mu}\, \delta^{\mu\nu} \underbrace{\delta_{ij}\, \delta_{ij}}_{\delta_{ii} = n_\mu} = [g]\, \delta^{\mu\nu}$$

$$\Rightarrow \frac{1}{[g]} \sum_{g \in G} \chi^{(\mu)}(g)\, \chi^{(\nu)}(g^{-1}) = \boxed{\frac{1}{[g]} \sum_{g \in G} \chi^{(\mu)}(g)\, \chi^{(\nu)*}(g) = \delta^{\mu\nu}} = \langle \chi^{(\mu)}, \chi^{(\nu)} \rangle$$

If we interpret $(\chi^{(\mu)}(g_1), \ldots, \chi^{(\mu)}(g_{[g]}))$ again as a vector and define a scalar product

φ, χ characters: $\langle \varphi, \chi \rangle = \frac{1}{[g]} \sum_g \varphi(g)\, \chi(g)^* = \langle \chi, \varphi \rangle^*$

two characters of different irreps are orthonormal!