$\Rightarrow$ The number of irreps is finite for finite groups G and is bounded by the dimension $[g]$ of the group G.

(we will show later that $\sum_\mu n_\mu^2 = [g]$).

Orthogonality of characters

- We had introduced the character of a matrix rep. D by the set of all its traces $\chi = \{\chi(g) = \mathrm{tr}\, D(g) \mid g \in G\}$. These have the following properties

(i) χ is the same for equivalent matrix reps D, D':
$D'(g) = S\, D(g)\, S^{-1} \; \forall g \in G$ (as we had already seen due to the cyclicity of the trace)

(ii) χ is the same for conjugate group elements:
$g, h \in G$: $D(hgh^{-1}) = D(h)\, D(g)\, D(h)^{-1}$, again use that trace is cyclic

(iii) if D is unitary, i.e. $D^{-1} = D^{+}$, then

$$\chi(g^{-1}) = \mathrm{Tr}\left(D(g)^{-1}\right) = \mathrm{Tr}\left(D(g)^{+}\right) = \chi(g)^{*}$$

The latter is thus always true for completely reducible reps as we may choose an equiv. unitary rep. Taking the trace over both matrices in the relation on p. 50 we thus have

$$\sum_{g \in G} \chi^{(\mu)}(g)\, \chi^{(\nu)}(g^{-1}) = \frac{[g]}{n_\mu}\, \delta^{\mu\nu} \underbrace{\delta_{ij}\,\delta_{ij}}_{\delta_{ii} = n_\mu} = [g]\, \delta^{\mu\nu}$$

$$\Rightarrow \frac{1}{[g]} \sum_{g \in G} \chi^{(\mu)}(g)\, \chi^{(\nu)}(g^{-1}) = \boxed{\frac{1}{[g]} \sum_{g \in G} \chi^{(\mu)}(g)\, \chi^{(\nu)*}(g) = \delta^{\mu\nu}} = \langle \chi^{(\mu)}, \chi^{(\nu)} \rangle$$

If we interpret $(\chi^{(\mu)}(g_1), \ldots, \chi^{(\mu)}(g_{[g]}))$ again as a vector and define a

scalar product φ, χ characters $\quad \langle \varphi, \chi \rangle = \frac{1}{[g]} \sum_g \varphi(g)\, \chi(g)^{*} = \langle \chi, \varphi \rangle^{*}$

two characters of different irreps are orthonormal!

- back to (ii): all elements in a conjugacy class have the same character & recall that the different conjugacy classes K_i partition a group into disjoint sets.

$\Rightarrow$ we may label the distinct characters χ_i, $i=1,\dots,k$ by the <u>number k of conjugacy</u> classes K_i.

- Let k_i be the number of elements in conjugacy class K_i

$\Rightarrow$ the orthogonality relation for characters can be rewritten as

$$\frac{1}{[g]} \sum_{i=1}^{k} k_i \chi_i^{(\mu)} \chi_i^{(\nu)*} = \delta^{\mu\nu}$$

$\Rightarrow$ the vectors $(k_i)^{\frac{1}{2}} \chi_i^{(\mu)} / [g]^{\frac{1}{2}}$ (no sum over i) are orthonormal, and there are no more than k of those

$\Rightarrow$ || The number of inequiv. irreps r is less or equal to the number of conjugacy classes k: $\underline{r \leq k}$

(note it can be shown that "=" holds, using $\frac{1}{[g]} \sum_{\mu \text{ irreps}} k_i \chi_i^{(\mu)} \chi_j^{(\mu)*} = \delta_{ij}$

see book of M. Hamermesh, Group Theory and its Applications to Physical Problems, Addison-Wesley, 1964)

An application of the orthogonality of group characters:

Decomposition of Reducible Reps:

- as we have seen for finite groups any reducible rep is completely reducible: $D(g) = \begin{pmatrix} A(g) & 0 \\ 0 & B(g) \end{pmatrix}$ $\forall g \in G$ (or $D(g) = A(g) \oplus B(g)$)

* after reducing (if possible) the blocks A & B further, the question arises: how often does an irrep $D^{(\nu)}$ arise in a given D?

so $D = \sum_{\oplus} a_\nu D^{(\nu)}$, what are the a_ν?

→ taking the trace we obtain:

$$\underbrace{\chi(g)}_{\text{compound character of } D} = \sum_\nu a_\nu \chi^{(\nu)}(g)$$

↑ simple characters of irreps, labelled by ν

- multiply with $\chi^{(\mu)}(g^{-1})$ and sum over g

$$\Rightarrow \sum_g \chi^{(\mu)}(g^{-1}) \chi(g) = \sum_\nu a_\nu \underbrace{\sum_g \overset{\text{"}\chi^{(\mu)}(g)^*\text{"}}{\chi^{(\mu)}(g^{-1})} \chi^{(\nu)}(g)}_{= [g]\,\delta^{\mu\nu}}$$

$$\Rightarrow \boxed{a_\mu = \frac{1}{[g]} \sum_g \chi(g) \chi^{(\mu)}(g^{-1})} = \langle \chi, \chi^{(\mu)} \rangle$$

The regular representation

We have shown in Caley's Theorem (p. 15) that an isomorph. exists between every finite group G of order $[g]$ and a subgroup of the permutations $S_{n=[g]}$ wrt left multiplication:

for a fixed $g \in G$ $\{g g_1, g g_2, \dots, g g_{[g]}\}$ is a permutation of the elements of G:

$$\forall i = 1, \dots, [g] \qquad g g_i = \sum_{j=1}^{[g]} D_{ji}(g)\, g_j$$ (mult. from the right)

where here $D_{ji}(g)$ is a permutation matrix of dim $[g] \times [g]$.
It is called the regular representation. It has only one 1 per each row & column.

For $g = e$ $D_{ji}(e) = \delta_{ji}$ is the identity matrix with only diagonal elements,

For $g \neq e$ $D_{ji}(g)$ has only off diagonal elements

example $C_3 = \{e, c, c^2\}$, $\qquad cC_3 = \{c, c^2, e\}$, $c^2C_3 = \{c^2, e, c\}$

$$D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D(c) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad D(c^2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Note that a special feature of the regular rep. is that its character is

$$\chi(g) = \begin{cases} [g] = \operatorname{Tr} D(e) \\ 0 = \operatorname{Tr} D(g \neq e) \quad \forall g \in G \end{cases}$$

we may use this feature to decompose the regular rep into its irreducible components:

$$D = \sum_{\oplus} a_\nu D^{(\nu)} \quad \text{with} \quad a_\mu = \frac{1}{[g]} \sum_{g \in G} \chi(g) \chi^{(\mu)}(g^{-1})$$

$$\Rightarrow a_\mu = \chi^{(\mu)}(e) = n_\mu \text{, the dim of the irrep } D^{(\mu)}$$

For $g = e$ in $\chi(g) = \sum_\nu a_\nu \chi^{(\nu)}(g)$ we obtain

$$\boxed{[g] = \sum_\nu n_\nu^2}$$

, so here we have "=" for p. 51

Note: for $g \neq e$: $0 = \sum_\nu n_\nu \chi^{(\nu)}(g) = \sum_\nu \chi^{(\nu)}(e) \chi^{(\nu)}(g)$

so in total $\frac{1}{[g]} \sum_\nu \chi^{(\nu)}(e) \chi^{(\nu)}(g) = \begin{cases} 0 & g \neq e \\ 1 & g = e \end{cases}$

which is a special case of p. 53 with $k_i = 1$

Characters of Irreps: the Character Table:

We have seen the following properties:

(1) the number of irreps r = the number of conjugacy classes k

(2) $\sum_{\mu \text{ irrep}} n_\mu^2 = [g]$ where n_μ is the dim of the irreps

(3) orthogonality $\sum_{i=1}^{k} k_i \chi_i^{(\mu)} \chi_i^{*(\mu)} = [g]$

where for (1) we only gave a proof for "≤"

→ we will list

	conj. classes $1 \dots k$
irreps 1	
$\vdots$	
r	

- in some cases additional info is available:

* for 1-dim reps. the characters and the 1x1 "matrix" rep are the same

* for Abelian groups all irreps are 1-dimensional :

proof Schur's first lemma:

take for $B = D^{(\mu)}(g)$ for any fixed $g \in G$

$\Rightarrow$ as G is Abelian $D^{(\mu)}(g) D^{(\mu)}(g') = D^{(\mu)}(g') D^{(\mu)}(g) \quad \forall g' \in G$

as $D^{(\mu)}(g)$ commutes with all other matrices in that rep

$\Rightarrow \quad D^{(\mu)}(g) = \lambda_g^{(\mu)} \mathbb{1}$, for all $g \in G$ as g was arbitrary

$\Rightarrow$ $D^{(\mu)}$ is reducible (to 1x1 blocks) (i.e. $\frac{1}{[g]} \sum_g$ exists)

* this can be extended to any compact Abelian group. As we have seen for G Abelian its conjugacy classes are all singletons, so $k = [g]$ and $K_i = \{g_i\}$ $i = 1, \dots, [g]$ are the conjugacy classes

$\Rightarrow \quad [g] = \sum_{\mu=1}^{r=k=[g]} n_\mu^2 \quad \Rightarrow \quad n_\mu^2 = 1 \;\; \forall \mu$ so all irreps are 1-dim

example C_3 : is Abelian, so its conj. classes are $\{e\}, \{c\}, \{c^2\}$

and $k = 3$

character table

C_3	e	c	c^2	
$D^{(1)}$	1	1	1	not faithful : $\ker = C_3$
$D^{(2)}$	1	ω	ω^2	faithful $\ker = \{e\}$
$D^{(3)}$	1	ω^2	ω	" " "

irreps, 1 dim