- In some cases additional info is available:
  * for 1-dim reps. the characters and the 1x1 "matrix" rep are the same
  * for Abelian groups all irreps are 1-dimensional :

proof Schur's first lemma:

take for $B = D^{(\mu)}(g)$ for any fixed $g \in G$

$\Rightarrow$ as G is Abelian $D^{(\mu)}(g)\, D^{(\mu)}(g') = D^{(\mu)}(g')\, D^{(\mu)}(g) \quad \forall g' \in G$

as $D^{(\mu)}(g)$ commutes with all other matrices in that rep

$\Rightarrow \quad D^{(\mu)}(g) = \lambda_g^{(\mu)} \mathbb{1}$, for all $g \in G$ as g was arbitrary.

$\Rightarrow \quad D^{(\mu)}$ is reducible (to 1x1 blocks) (i.e. $\frac{1}{[g]} \sum_g$ exists)

* this can be extended to any compact Abelian group. As we have seen for G Abelian its conjugacy classes are all singletons, so $k = [g]$ and $K_i = \{g_i\}$ $i = 1, \dots, [g]$ are the conjugacy classes

$\Rightarrow \quad [g] = \sum_{\mu=1}^{r=k=[g]} n_\mu^2 \quad \Rightarrow \quad n_\mu^2 = 1 \;\; \forall \mu$ so all irreps are 1-dim

example C_3 : is Abelian, so its conj. classes are $\{e\}, \{c\}, \{c^2\}$ (3)

and $k = 3$

character table

C_3	e	c	c^2	
$D^{(1)}$	1	1	1	not faithful : Kernel $= C_3$
$D^{(2)}$	1	ω	ω^2	faithful Kernel $= \{e\}$
$D^{(3)}$	1	ω^2	ω	" " "

irreps, 1 dim

- the characters (traces) and 1x1 dim matrix reps are the same

$\Rightarrow$ the characters themselves have to obey the group multiplication:

$\chi(c^2) = (\chi(c))^2$ and $(\chi(c))^3 = \chi(c^3) = \chi(e) = 1$

$\Rightarrow$ $\chi(c)$ must be a root of unity, here $1, \omega = e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}} = \omega^2$

- $D^{(1)}$ is the trivial rep
- $D^{(2)}$ and $D^{(3)}$ are complex conjugate as $\omega^* = e^{-\frac{2\pi i}{3}} = e^{-\frac{2\pi i}{3} + \frac{6\pi i}{3}} = \omega^2$

for all 3 reps we have a homomorphism $G \to GL(1, \mathbb{C})$

- as C_3 is Abelian it has no proper subgroups, only $\{e\}$ and C_3 are subgroups
- recall the special relation of normal subgroups ($H \leq G$ subgroup, left = right cosets always for Abelian) and conjugacy classes (labeling irreps):

- a normal subgroup is made of complete conjugacy classes (p. 31D) (p. 28)
- because the kernel* is a normal subgroup itself ($= \{e\}$ or C_3) we can distinguish irreps by their kernel:

$D^{(1)}$: kernel $K = C_3$, not faithful

$D^{(2)}$ & $D^{(3)}$ kernel $K = \{e\}$ so both are faithful

* $D : G \to GL(1, \mathbb{C})$ ($= \mathbb{C} \setminus \{0\}$) is a map from group G to the group $GL(1,\mathbb{C})$ (with multiplication on $\mathbb{C}$, neutral element 1)

Check of orthonormality:

$$\frac{1}{[g]} \cdot \sum_{i=1}^{3=k=[g]} \underset{\overset{\shortparallel}{1}}{k_i} \chi_i^{(\mu)} \chi_i^{(\nu)*} = \delta^{\mu\nu}$$

$\underline{\mu=\nu:}$ D^1 ✓, $D^{2,3}$ ✓ as $(\omega\omega^*)^{1,2} = 1$

$\underline{\mu\neq\nu:}$ we need to show a) $\boxed{1+\omega+\omega^2 = 0}$ for $\omega\neq 1$
$(\nu=1, \mu=2,3)$

which is true for roots of unity $\omega^3=1$

$$\Rightarrow (\omega-1)(1+\omega+\omega^2) = 0$$

b) $\boxed{1+\omega\omega^{*2}+\omega^2\omega^* = 1+\underset{\overset{\shortparallel}{e^{-\frac{2\pi}{3}i} = e^{\frac{6\pi}{3}i-\frac{2\pi}{3}i} = \omega^2}}{\omega^*}+\omega = 0} ✓$
$(\nu=2, \mu=3)$

Direct Products of Representations and Their Decomposition

- So far we have decomposed reducible reps D into their irreps, e.g. $D(g) = A(g) \oplus B(g)$ or more general $D = \sum_{\oplus} a_\nu D^{(\nu)}$
- Q: Can we also study products of reps?

Reminder - so far we have studied G itself as a direct product of subgroups A, B (p. 31A)

$G = A\times B$ if $\forall a\in A\ \forall b\in B$ $ab=ba$ and $\forall g\in G\ \exists! a\in A\ \exists! b\in B$ s.th. $g=ab$

(or products of different groups: Cartesian product)

Now we'd like to study also products of reps of the same group G (not only subgroups)

Example: Consider a qm. 2-particle system, e.g. 2 electrons of 2 atoms on a crystal (lattice) or of free atoms. These el. are described by spatial wave functions $\psi_a(x)$, $\varphi_c(x)$.

Suppose we have a symmetry group G acting on the system, and (lattice sym. or cont. rotation sym.) $\psi_a(x)$ and $\varphi_c(x)$ transform wrt the two irrep $D^{(\mu)}, D^{(\nu)}$: $g \in G$

$$\psi'_a(x) = D^{(\mu)}_{ab}(g)\psi_b(x) \qquad \Psi_{ac}(x) \equiv \psi_a(x)\varphi_c(x) \quad \text{product wave fct.}$$

$$\varphi'_c(x) = D^{(\nu)}_{cd}(g)\varphi_d(x) \quad \text{, or} \quad \text{with } \Psi'_{ac}(x) = \underbrace{D^{(\mu)}_{ab}(g)D^{(\nu)}_{cd}(g)}_{\equiv D^{(\mu\times\nu)}_{AB}(g)}\Psi_{bd}(x)$$

We can regard the product of the 2 matrices as a single matrix with indices $A = (ac)$ (ordered pair) and $B = (bd)$: $\Psi'_A(x) = D^{(\mu\times\nu)}_{AB}\Psi_B$

The rep $D^{(\mu\times\nu)}_{AB}(g)$ is then called a direct product rep of $\boxed{D^{(\mu)} \otimes D^{(\nu)}}$. We have to check that it is a rep itself:

$$D^{(\mu\times\nu)}(g_1)D^{(\mu\times\nu)}(g_2) = D^{(\mu)}(g_1)D^{(\nu)}(g_1)D^{(\mu)}(g_2)D^{(\nu)}(g_2) = D^{(\mu)}(g_1g_2)D^{(\nu)}(g_1g_2) = D^{(\mu\times\nu)}(g_1g_2) \checkmark$$

Q: is $D^{(\mu\times\nu)}$ also an irrep (as were $D^{(\mu)}$ and $D^{(\nu)}$)?

1) the electrons don't interact:

$\Rightarrow$ acting on ψ with g_1 and on φ with g_2 is a symmetry, provided by $D^{(\mu\times\nu)}_{ac,bd}(g_1, g_2) \equiv D^{(\mu)}_{ab}(g_1)D^{(\nu)}_{cd}(g_2)$

This is an irrep of the product Group $G \times G$

(assume that not: $\exists$ inv. proper subspace $\forall g_1, g_2 \in G$, in particular for $g_2 = e \Rightarrow \exists$ " " of $D^{(\mu)}$ ↯)

2) the electrons do interact

$\Rightarrow$ we only have a residual invariance taking $g_1 = g_2 = g$

In general then $D^{(\mu)} \otimes D^{(\nu)}$ is reducible:

$$D^{(\mu)} \otimes D^{(\nu)} = \sum_{\oplus} a_\sigma D^{(\sigma)}$$ called Clebsch-Gordan series

- the a_σ can be determined using the orthogonal simple characters:

we have $\chi^{(\mu\times\nu)}(g) = D^{(\mu\times\nu)}_{AA}(g) = D^{(\mu)}_{aa}(g)\, D^{(\nu)}_{cc}(g) = \chi^{(\mu)}(g)\chi^{(\nu)}(g)$

p. 54

$$\Rightarrow a_\sigma = \frac{1}{[g]} \sum_{g \in G} \chi^{(\mu)}(g)\chi^{(\nu)}(g)\, \chi^{(\sigma)}(g^{-1}) = \langle \chi^{(\mu)}\chi^{(\nu)}, \chi^{(\sigma)} \rangle$$

Continuous Groups

Apart from some examples we have so far only dealt with discrete groups, i.e. rotations by multiples of a fixed angle $\frac{2\pi}{n}$, $n \in \mathbb{N}$.

For obvious reasons in physics rotations wrt a continuous angle play an important rôle. In general we would like to treat the situation where group elements are characterised by n real variables, that is some $x \in \mathbb{R}^n$. For group operations we then have for $g(x) \in G$

product $\forall x, y \in \mathbb{R}^n \; \exists z \in \mathbb{R}^n$ s.th. $\boxed{g(x) \cdot g(y) = g(z)}$ with $z = f_1(x,y)$

with $f_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$

inverse $\forall x \in \mathbb{R}^n \; \exists w \in \mathbb{R}^n$ s.th. $g(x)^{-1} = g(w)$

with $f_2(x) = w$, $f_2 : \mathbb{R}^n \to \mathbb{R}^n$

Def: A group G is called continuous if the above applies and f_1 and f_2 are continuous

Def If in addition f_1 and f_2 are analytic ($\in C^\infty$, expandible in a Power-Taylor series) then G is called a Lie group.

Example:

The matrixrep on p. 32 $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ of rotations in $\mathbb{R}^3$ around the z-axis parametrized by an angle $\theta \in [0, 2\pi)$ is a Lie group. It is also called a compact group since $\theta \in [0, 2\pi) \subseteq \mathbb{R}$ which is a set that is: bounded & single component (not closed)

Examples:

- we have already seen that linear maps / matrix reps that preserve the scalar product are unitary. How many real parameters do they contain?

compl. scalar product on $\mathbb{C}^n$:

$\boxed{U(n)} \subseteq GL(n, \mathbb{C})$ mit $U^{-1}_{ij} = U^{+}_{ij}$ or $UU^{+} = \mathbb{1}_{n\times n}$

- has n^2 complex, so $2n^2$ real entries, plus constraints to satisfy

$$U_{i\ell} U^{+}_{\ell j} = U_{i\ell} U^{*}_{j\ell} = \delta_{ij}$$

- $i = j$: n real eqs.
- $i < j$: $\frac{n(n-1)}{2}$ complex eqs

($i > j$ gives the same eqs.)

$\Rightarrow$ we have

$$2n^2 - n - 2 \cdot \frac{n(n-1)}{2} = n^2 \text{ real parameters}$$

$\boxed{SU(n)}$ has in addition $\det U = +1$:

1 real eq. ($\det U^{*} = \det U^{T^{-1}} = \det U^{-1} = \frac{1}{\det U} = 1$ is the same)

$\Rightarrow$ $n^2 - 1$ real parameters