• in case we restrict ourselves to real matrix elements (and to a real scalar product) we have that

$O(n) \subseteq GL(n, \mathbb{R})$ are unitary $O^{-1} = O^{+} = O^{T}$

so also orthogonal $O O^{T} = \mathbb{1}_{n\times n} \iff O_{i\ell} O_{j\ell} = \delta_{ij}$

$\Rightarrow$ we have n^2 real elements $-$ n eqs for $i=j$

$- \frac{n(n-1)}{2}$ eqs. for $i<j$

$\Rightarrow n^2 - n - \frac{n(n-1)}{2} = \frac{n^2-n}{2} = \frac{n(n-1)}{2}$ real parameters

$SO(n)$ has in addition $\det O = 1$ 1 real eqs.:

but $\det O = \det O^{T} \Rightarrow 1 = \det \mathbb{1} = \det O O^{T} = (\det O)^2$

$\Rightarrow \det O = \pm 1$ $\exists$ 2 components, those with $+1$ or -1

still have $\frac{n(n-1)}{2}$ real parameters (for $U(n)$ we'd get $|\det U|^2 = 1$, so phase, get real constraint)

* let us consider 2 examples for this which are very important in physics: $SO(2)$ and $SU(2)$. These particular cases enjoy a relation too

Def: The n-sphere S^n is defined as the following subset of $\mathbb{R}^{n+1}$, $n \geq 1$: $S^n = \{ \vec{x} \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = \vec{x}^T \cdot \vec{x} = 1 \}$

$n=1$  $n=2$ 

• example $SO(2)$:

these are all invertible 2×2 matrices with real elements

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ that satisfy $A^T = A^{-1}$ and $\det A = 1$

• from our previous count this has $\frac{n(n-1)}{2} = \frac{2\cdot 1}{2} = 1$ real parameter:

$$A^T A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\Leftrightarrow$ 3 eqs: i) $a^2+c^2=1$, ii) $b^2+d^2=1$, iii) $ab+cd=0$

1) distinguish $a \neq 0$: $\Rightarrow$ iii) $b = -\frac{cd}{a}$

in ii) $\frac{c^2d^2}{a^2} + d^2 = 1 \Leftrightarrow (c^2+a^2)d^2 = a^2$ ($\cdot a^2 \neq 0$), with $c^2+a^2 = 1$ i)

$\Rightarrow d^2 = a^2 \Rightarrow d = \pm a \Rightarrow b = \mp c$

$O(2) \ni$

$$\Rightarrow \boxed{A = \begin{pmatrix} a & b \\ \mp b & \pm a \end{pmatrix}}$$

we still need to impose $\det A \stackrel{!}{=} 1$

$\Rightarrow \det A = \pm a^2 - (\mp b^2) = \pm(a^2+b^2) = 1 \Rightarrow$ only choice $(+)$

$a = 0$: $\Rightarrow c^2 = 1 \Leftrightarrow c = \pm 1 \Rightarrow 0\cdot b + (\pm 1)d = 0 \Rightarrow d = 0$

$\Rightarrow b^2 = 1 \Rightarrow b = \pm 1$

$\Rightarrow$ 4 possibilities $A = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$

$\det A = 1 = -bc \Rightarrow$ we can only have $\begin{cases} c = +1 \\ b = -1 \\ \quad = -c \end{cases}$ or $\begin{cases} c = -1 \\ b = +1 \\ \quad = -c \end{cases}$

so $\boxed{A = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}}$ with $b^2 = +1$

• in other words: $\det A = 1$ only picks a sign and does not eliminate a further real variable

in total $\boxed{A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \text{with } a^2+b^2 = 1}$

$\Rightarrow$ the elements of S^1 parametrise $SO(2)$, thus it is simply connected

* group isomorphism $SO(2) \cong U(1)$:

we had already seen a solution to this parametrisation of SO(2).

$a = \cos(\theta)$, $b = -\sin(\theta)$, $\theta \in [0, 2\pi]$

$\Rightarrow A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ with $\cos^2\theta + \sin^2\theta = 1$ ✓

. on the other hand the group $U(1) \in GL(1, \mathbb{C})$ has $n^2 = 1$ real param:

these are 1x1 complex "matrices" $z \in \mathbb{C}$ with $z^* = z^{-1}$:

$z z^* = 1$, or in polar coordinates $z = r e^{i\theta}$ with $r = 1$

$= |z|^2$ $\qquad\qquad = \cos\theta + i \sin\theta$

$\Rightarrow$ we have established a 1-1 corresp. between SO(2) and U(1)

. both groups are Abelian so their (complex) irreps are 1-dim.

these 1d irreps satisfy $D(\varphi) = D(\varphi + 2\pi)$: they can be labelled by an integer $m \in \mathbb{Z}$: $D^{(m)}(\varphi) = e^{-im\varphi}$

$\rightarrow$ this includes an unfaithful irrep. for $m = 0$: $D^{(0)}(\varphi) = 1$

(Note that in quantum mechanics also reps with $D(\varphi) = -D(\varphi + 2\pi)$ are allowed $\rightarrow$ Fermions)

• example SU(2):

= all invertible 2x2 matrices with complex elements

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in \mathbb{C}$ and $A^\dagger = A^{-1}$ satisfying $\det A = +1$

$\Rightarrow A^\dagger A = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} |a|^2 + |c|^2 & a^* b + c^* d \\ a b^* + c d^* & |b|^2 + |d|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

* Let's apply what we have learned to the example $SO(2) \cong U(1)$:

- the irreps $D^{(m)}(\varphi) = e^{-im\varphi}$ are 1dim and thus equal to the characters: $\mathrm{Tr}\, D^{(m)}(\varphi) = \chi^{(m)}(\varphi) = e^{-im\varphi}$

- the orthogonality of characters wrt $\frac{1}{[g]}\sum_g$ can be generalized to continuous groups that are compact: $\rightarrow \frac{1}{2\pi}\int_0^{2\pi} d\varphi$, as $\varphi \in [0, 2\pi)$

$$\Rightarrow \langle \chi^{(m)}, \chi^{(m')} \rangle = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, e^{im\varphi} e^{-im'\varphi} = \delta_{mm'} \quad ✓$$

- the <u>Clebsch-Gordan series for U(1)</u> is particularly simple:

$$D^{(m)} \otimes D^{(m')} = D^{(m+m')}$$ which is again an irrep.

- the 2-dim rep $A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ was reducible (being 2-dim)

and its Clebsch-Gordan coeffs are easily obtained:

$\mathrm{Tr}\, A = 2\cos\varphi \quad (= \mathrm{Tr}\, A^*)$

$$\Rightarrow \quad a_m = \langle \chi^{(m)}, \chi \rangle = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, e^{im\varphi}\, 2\cos\varphi\, d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, e^{im\varphi}\left(e^{i\varphi} + e^{-i\varphi}\right)$$

$$= \delta_{m,-1} + \delta_{m,1}$$

this should lead to $n^2-1 = 3$ real elements:

3 real eqs: i) $|a|^2+|c|^2=1$ ii) $|b|^2+|d|^2=1$, iv) $ad-cb=1$

1 complex eq: iii) $a^*b+c^*d=0$

we will again consider 2 cases:

$a\neq 0$: iii) $b=-\frac{c^*d}{a^*}$ => iv) $ad+\frac{cc^*d}{a^*}=\frac{d}{a^*}\underbrace{(|a|^2+|c|^2)}_{i)\,=\,1}=\frac{d}{a^*}=1$

=> $d=a^*$, $b=-c^*$ and $A=\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$, $|a|^2+|b|^2=1$

$a=0$: i): $|c|^2=1$, iii) $c^*d=0$ => $d=0$ => ii) $|b|^2=1$

iv): $-cb=1 \quad |\cdot b^*$ => $-c\underbrace{|b|^2}_{1}=b^*$

=> $A=\begin{pmatrix} 0 & b \\ -b^* & 0 \end{pmatrix}$ with $|b|^2=1$

=> we can write $a=x_1+ix_2$, $b=x_3+ix_4$ with

$$A=\begin{pmatrix} x_1+ix_2 & x_3+ix_4 \\ -x_3+ix_4 & x_1-ix_2 \end{pmatrix} \text{ with } \sum_{i=1}^{4} x_i^2 = 1$$

so the elements of S^3 parametrise $SU(2)$

* Note: S^1 and S^3 are the only n-spheres that represent groups = group manifolds. Other geom. objects that fulfil this are tori T^n

just as in the isomorphism $SO(2)\cong U(1)$ it would be nice to have a representation of $SU(2)$ that contains only 3 real parameters explicitly, and not 4 with a constraint. This can be achieved with the

Def: Pauli matrices $\sigma_1=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$