consider a 3-dim unit vector $\hat{n} = (n_1, n_2, n_3)$ with $|\hat{n}| = 1$

$$\Rightarrow (n_i \sigma_i)^2 = \begin{pmatrix} n_3 & n_1 - i n_2 \\ n_1 + i n_2 & -n_3 \end{pmatrix}^2 = \begin{pmatrix} n_3^2 + (n_1 - i n_2)(n_1 + i n_2) & 0 \\ 0 & (n_1 - i n_2)(n_1 + i n_2) + n_3^2 \end{pmatrix}$$

$$= (n_1^2 + n_2^2 + n_3^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}_{2\times 2}$$

$$\Rightarrow (n_i \sigma_i)^k = \begin{cases} \hat{n}_i \sigma_i & k \text{ odd} \\ \mathbb{1}_{2\times 2} & k \text{ even} \end{cases}$$

claim: we can rep. A as follows for $A \in SU(2)$

$$\boxed{A = \exp\left(-\frac{i}{2} \theta_a \sigma_a\right)} = \exp\left(-\frac{i}{2} \theta \, \hat{n}_a \sigma_a\right) \quad \text{with} \quad \theta_a \in [0, 2\pi] \; a = 1,2,3$$

$$\theta = \theta_a \theta_a$$

$$\hat{n} = \frac{\vec{\theta}}{\theta} \text{ for } \theta \neq 0$$

$$= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left(\frac{-i\theta}{2}\right)^\ell (\hat{n}_a \sigma_a)^\ell$$

$$= \sum_{k=0}^{\infty} \frac{1}{(2k)!} \frac{(-i)^{2k} \theta^{2k}}{2^{2k}} \mathbb{1}_{2\times 2} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \frac{(-i)^{2k+1}}{2^{2k+1}} \theta^{2k+1} \hat{n}_a \sigma_a$$

$$= \cos\left(\frac{\theta}{2}\right) \mathbb{1}_{2\times 2} - i \sin\left(\frac{\theta}{2}\right) (\hat{n}_i \sigma_i)$$

$$= \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) - i n_3 \sin\left(\frac{\theta}{2}\right) & -(i n_1 + n_2) \sin\left(\frac{\theta}{2}\right) \\ -(i n_1 - n_2) \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) + i n_3 \sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$\equiv \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad \text{with } aa^* + bb^* = \cos^2\left(\frac{\theta}{2}\right) + n_3^2 \sin^2\left(\frac{\theta}{2}\right) + (n_1^2 + n_2^2) \sin^2\left(\frac{\theta}{2}\right)$$

$$= \cos^2\left(\frac{\theta}{2}\right) + \underbrace{(n_1^2 + n_2^2 + n_3^2)}_{1} \sin^2\left(\frac{\theta}{2}\right) = 1$$

* This case is exemplary for Lie-groups: we have the group parametrised by some generators – here $\sigma_{i=1,2,3}$. They contain all info about the group and satisfy some commutation relations – here $[\sigma_a, \sigma_b] = 2i\varepsilon_{abc}\sigma_c$, the algebra of this group.

Lie - Groups and Lie - Algebras

- for continuous (Lie!) groups we need a few more concepts:

Def: The dimension d (or dim) of a continuous group is given by the number of real parameters parametrising the group

Examples: $\dim[SO(n)] = \frac{n(n-1)}{2}$

$\dim[SU(n)] = n^2 - 1$

- In general Lie-groups will have several disconnected components. In the following we denote by $\mathcal{H}$ the connected component that contains the identity.

Example: $O(2)$ has 2 components, with $\det A_1 = +1 = -\det A_2$

$$A_1 = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \quad \text{both with } a^2+b^2=1$$

Only the first contains $\mathbb{1}_{2\times2}$: $a=1$, $b=0$

- we have seen that unitary maps T (matrix reps. D) preserve the scalar product: $\forall v, w \in \mathbb{C}^n$: $(Tv, Tw) = (v, w)$

in case the scalar product is defined by a metric η,

$(v, w) = v_i^* \eta_{ij} w_j$ this implies

$= (Dv, Dw) = D_{ij}^* v_j^* \eta_{ik} D_{kl} w_l \Rightarrow D_{ji}^T \eta_{ik} D_{kl} = \delta_{jl}$

So the matrix rep leaves the metric invariant:

$$D^+ \eta D = \eta \Leftrightarrow D^+ \eta = \eta D^+$$

The following Theorem holds for Lie-groups.

Any unitary matrix rep $D : G \to GL(n, \mathbb{C})$ with $D \in \mathcal{K}$ can be written as

$$\boxed{D = \exp\left(i \sum_{a=1}^{\dim[G]} \theta^a T^a\right)}$$

where exp is defined by its Taylor expansion. The nxn matrices T^a $a=1, \dots, \dim$ are called the generators of Lie-group G and have the following properties:

$\forall a\, (T^a)^\dagger \eta - \eta T^a = 0 \iff T^a = T^{a\dagger}$. If in addition $\det D = 1$ is imposed then $\mathrm{Tr}(T^a) = 0$ $\forall a$. Here the parameters θ^a $a=1, \dots, \dim[G]$ are real.

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we won't give a complete proof here. The properties of T^a follow from

- $\eta = D^\dagger \eta D = (1 - i\theta^a T^{a\dagger})\, \eta\, (1 + i\theta^a T^a)$ for infinitesimal θ^a's

$$= \eta - i\theta^a \underbrace{(T^{a\dagger}\eta - \eta T^a)}_{0} + \mathcal{O}(\theta^{a2}) \overset{\text{p.63}}{\Longrightarrow} T^a = T^{a\dagger}$$

- we have for $1 = \det D = \exp[\mathrm{Tr} \log \exp(i\theta^a T^a)]$

$$= \exp[i\theta^a \mathrm{Tr}\, T^a] \overset{!}{=} 1$$

$\Rightarrow$ $\forall a\, \mathrm{Tr}(T^a) = 0$ as the parameters θ^a can be chosen independently

- for $\theta^a = 0$ obviously $\mathbb{1}_{n\times n} \in \mathcal{K}$. Are all D above connected to the identity in $\mathcal{K}$? Let $\theta^a T^a \equiv \theta$ with T^a as above, $\lambda \in [0,1]$ define $f(\lambda) = \exp[i\lambda\theta]$, $c(\lambda) = f(\lambda)^\dagger \eta f(\lambda)$

$$\Rightarrow \frac{d}{d\lambda} c(\lambda) \underset{\substack{\uparrow \\ \text{Lie-group is diff'ble}}}{=} \frac{d}{d\lambda}\left(e^{-i\lambda\theta} \eta e^{+i\lambda\theta}\right) = -ie^{-i\lambda\theta}(\theta\eta - \eta\theta)e^{i\lambda\theta} \underset{\substack{\uparrow \\ \text{prop. of } T^a}}{=} 0$$

$\Rightarrow$ $c(\lambda)$ is constant: $c(\lambda) = c(0) = \eta$ $\Rightarrow$ $f(\lambda)$ is a unitary matrix rep of G that is connected to $\mathbb{1}$ $\forall \lambda$, and $f(\lambda = 1)$ is our D above

* the difficult part of the proof is to show that the exponential rep. above is surjective, i.e. all elements in $\mathcal{G}$ can be written this way.

* in the following it will often be much simpler to analyse the matrices Θ spanned by the generators, rather than the matrix reps. D of the group itself.

Def the matrices $\Theta = \theta^a T^a$ with T^a the generators of Lie-group G constitute a vector space $\mathfrak{g}$ that we call Lie-algebra

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Notation: Lie-group $SU(2)$ $\to$ Lie-algebra $su(2)$

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* the group multiplication on G induces a multiplication law on $\mathfrak{g}$: for $A = \exp(i\,\theta^a T^a)$, $B = \exp(i\,\varphi^a T^a)$ we have

$$\boxed{A \cdot B = C = \exp(i\,\xi^a T^a)} \in G$$

- let us consider infinitesimal parameters $\theta^a, \varphi^a, \xi^a \ll 1$ and expand the non-commutative matrix products

* symmetrization and anti-...

Def $\{T^a, T^b\} = T^a T^b + T^b T^a$ anti-commutator

$[T^a, T^b] = T^a T^b - T^b T^a$ commutator

$$\Rightarrow \quad T^a T^b = \tfrac{1}{2}\{T^a, T^b\} + \tfrac{1}{2}[T^a, T^b] \qquad \text{(sym + anti-sym part)}$$

$$\xi^a \xi^b T^a T^b = \tfrac{1}{2}\xi^a \xi^b \{T^a, T^b\} \qquad \text{as } \xi^a \xi^b = \xi^b \xi^a \text{ commute}$$

$$\underbrace{\theta^a \varphi^b \{T^a, T^b\} = \tfrac{1}{2}(\theta^a \varphi^b + \theta^b \varphi^a)\{T^a, T^b\}}_{\text{sym}}$$