* the difficult part of the proof is to show that the exponential rep. above is surjective, i.e. all elements in $\mathcal{G}$ can be written this way.

* in the following it will often be much simpler to analyse the matrices θ spanned by the generators, rather than the matrix reps. D of the group itself.

<u>Def</u> the matrices $\Theta = \theta^a T^a$ with T^a the generators of Lie-group G constitute a vector space $\mathfrak{g}$ that we call <u>Lie-algebra</u>

Notation: Lie-group $SU(2)$ $\to$ Lie-algebra $su(2)$

---

* the <u>group multiplication</u> on G induces a <u>multiplication law</u> <u>on $\mathfrak{g}$:</u> for $A = \exp(i\,\theta^a T^a)$, $B = \exp(i\,\varphi^a T^a)$

we have $\boxed{A \cdot B = C \equiv \exp(i\,\xi^a T^a)}$ $\in G$

- let us consider infinitesimal parameters $\theta^a, \varphi^a, \xi^a \ll 1$ and expand the non-commutative matrix products

* <u>symmetrization and ~~anti-~~</u>

<u>Def</u> $\{T^a, T^b\} \equiv T^a T^b + T^b T^a$ <u>anti-commutator</u>

$[T^a, T^b] \equiv T^a T^b - T^b T^a$ <u>commutator</u>

$$\Rightarrow \quad T^a T^b = \frac{1}{2}\{T^a, T^b\} + \frac{1}{2}[T^a, b^b] \quad \text{(sym + anti-sym part)}$$

$$\xi^a \xi^b T^a T^b = \frac{1}{2}\xi^a \xi^b \{T^a, T^b\} \quad \text{as } \xi^a \xi^b = \xi^b \xi^a \text{ commute}$$

$$\theta^a \varphi^b \underbrace{\{T^a T^b\}}_{\text{sym}} = \frac{1}{2}(\theta^a \varphi^b + \theta^b \varphi^a)\{T^a T^b\}$$

$\Rightarrow AB = C \Leftrightarrow$

$$\left[\mathbb{1} + i\theta^a T^a - \tfrac{1}{2}\theta^a\theta^b T^a T^b\right]\left[\mathbb{1} + i\varphi^a T^a - \tfrac{1}{2}\varphi^a\varphi^b T^a T^b\right] + \mathcal{O}(\varphi^3, \theta^3)$$

$$= \mathbb{1} + i(\theta^a + \varphi^a) T^a - \tfrac{1}{2}(\theta^a\theta^b + \varphi^a\varphi^b + 2\theta^a\varphi^b) T^a T^b + \mathcal{O}(\varphi^3, \theta^3)$$

$$= \mathbb{1} + i(\theta^a + \varphi^a) T^a - \tfrac{1}{4}(\theta^a + \varphi^a)(\theta^b + \varphi^b)\{T^a, T^b\} - \tfrac{1}{2}\theta^a\varphi^b [T^a, T^b] + \mathcal{O}(\varphi^3, \theta^3)$$

$$\stackrel{!}{=} \mathbb{1} + i\xi^a T^a - \tfrac{1}{2}\underbrace{\xi^a\xi^b T^a T^b}_{\frac{1}{2}\xi^a\xi^b\{T^a, T^b\}}$$

- the symmetric and anti-sym part in a, b on the LHS & RHS have to be satisfied independently. This only works if we have the

$\boxed{[T^a, T^b] = i f^{abc} T^c}$ proportionality

where $f^{abc} = -f^{bac}$ are called <u>structure constants</u> of $\mathfrak{g}$.

$$\Rightarrow i(\theta^a + \varphi^a - \tfrac{1}{2}\theta^d\varphi^e f^{dea}) T^a = i\xi^a T^a + \mathcal{O}(\varphi^3, \theta^3, \xi^3)$$

$$\Leftrightarrow \boxed{\xi^a = \theta^a + \varphi^a - f^{dea}\theta^d\varphi^e} + \mathcal{O}(\xi^3, \theta^3, \varphi^3)$$

which satisfies $\{T^a, T^b\}\xi^a\xi^b = (\theta^a + \varphi^a - \tfrac{1}{2}f^{dea}\theta^d\varphi^e)(\theta^b + \varphi^b - \tfrac{1}{2}f^{fgb}\theta^f\varphi^g)\{T^a, T^b\}$

$$= (\theta^a + \varphi^a)(\theta^b + \varphi^b)\{T^a, T^b\} \text{ up to } \mathcal{O}(\xi^3, \theta^3, \varphi^3)$$

we define a <u>multiplication</u> on $\mathfrak{g}$

by: $\theta, \varphi \in \mathfrak{g} \Rightarrow \begin{cases} \theta \cdot \varphi \equiv [\theta, \varphi] = i\theta^a\varphi^b f^{abc} T^c \in \mathfrak{g} \\ \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \end{cases}$

Examples for generators and their algebra:

SO(2): recalling that we had (p. 65)

$$A(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

we can either obtain the T^a by expanding in θ for $\theta \ll 1$ (or by differentiation)

$$= \begin{pmatrix} 1-\frac{1}{2}\theta^2 & -\theta \\ \theta & 1-\frac{1}{2}\theta^2 \end{pmatrix} + \mathcal{O}(\theta^3) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} + \mathcal{O}(\theta^2)$$

$$= \mathbb{1} + i\theta^a T^a + \mathcal{O}(\theta^2)$$ [here $a = 1 = \dim[SO(2)]$]

$$\Rightarrow iT = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Leftrightarrow T = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -\sigma_2$$

which is Hermitian and $\mathrm{Tr}(T)=0$ due to $\det A = 1$ (and unitary: $T^2 = \mathbb{1}$)

(or $\left.\frac{dA}{d\theta}\right|_{\theta=0} = \left.\frac{d}{d\theta} e^{i\theta T}\right|_{\theta=0} = iT = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$)

- the group $SO(2)$ is Abelian and we have only 1 generator T, so the algebra $[T,T]=0$ is trivial
- We can easily rewrite $A(\theta)$ in exponential form ($SO(2)$ has only 1 comp = $\mathbb{K}$)

using $(iT)^k = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^k = \begin{cases} (-)^{\frac{k}{2}} \mathbb{1} & \text{for } k \text{ even} \\ (-)^{\frac{k-1}{2}} iT & \text{for } k \text{ odd} \end{cases}$

we have $A(\theta) \overset{!}{=} \exp[\theta iT] = \sum_{k=0}^{\infty} \frac{\theta^k (iT)^k}{k!} = \sum_{\ell=0}^{\infty} \frac{\theta^{2\ell}}{\boxed{2\ell}!} (-)^\ell \mathbb{1} + \sum_{\ell=0}^{\infty} \frac{\theta^{2\ell+1}}{(2\ell+1)!} (-)^\ell iT$

$$= \cos\theta + iT\sin\theta$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \checkmark$$

SO(3): This is the group of proper ($\det A = 1$) rotations in $\mathbb{R}^3$

- we still have to find an explicit parametrisation with

$$\frac{n(n-1)}{2} = \frac{3\cdot 2}{2} = 3 \text{ real variables:}$$

- the matrices representing rotations around the 3 Cartesian axes and their (infinitesimal) generators are easily written down:

z-axis
$$R_3(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

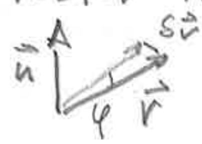
$$iT_3 = \left.\frac{dR_3}{d\varphi}\right|_{\varphi=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\varepsilon_{kl3} \Leftrightarrow \underline{(T_3)_{kl} = i\varepsilon_{kl3}}$$

x-axis
$$R_1(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}, \quad iT_1 = \left.\frac{dR_1}{d\varphi}\right|_{\varphi=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \Leftrightarrow \underline{(T_1)_{kl} = i\varepsilon_{kl1}}$$

y-axis
$$R_2(\varphi) = \begin{pmatrix} \cos\varphi & 0 & +\sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix}, \quad iT_2 = \left.\frac{dR_2}{d\varphi}\right|_{\varphi=0} = \begin{pmatrix} 0 & 0 & +1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \Leftrightarrow \underline{(T_2)_{kl} = i\varepsilon_{kl2}}$$

we can easily convince ourselves that the $T_{1,2,3}$ form a

- basis for infinitesimal rotations around unit vector $\vec{n}$ by φ:



$$\vec{r}\,' = \vec{r} + \delta\vec{r}, \quad \text{with } \delta\vec{r} = \vec{n} \times \vec{r}\cdot\varphi$$

$$\Leftrightarrow r'_l = r_l + \varphi\,\varepsilon_{lkj}\, n_k r_j = \underbrace{(\delta_{lj} + i\varphi\, i\varepsilon_{ljk} n_k)}\, r_j$$

$$\Leftrightarrow \underline{(R(\varphi))_{lj} = \delta_{lj} + i\varphi (T_k)_{lj} n_k}$$; we can also define the vector $\vec{\varphi} = \varphi\vec{n} = (\varphi n_1, \varphi n_2, \varphi n_3)$

- finite rotations

$$\boxed{R(\varphi) = \exp[i\varphi^k T_k]}$$

- The generators $T_{a=1,2,3}$ satisfy the following commutation relation $\boxed{[T_a, T_b] = -i\varepsilon_{abc} T_c}$ $so(3)$

RHS $-i\varepsilon_{abc}(T_c)_{ke} = -i^2 \varepsilon_{kec}\varepsilon_{abc}$ and

LHS $= (T_a T_b)_{ke} - (T_b T_a)_{ke} = (T_a)_{kj}(T_b)_{je} - (T_b)_{kj}(T_a)_{je}$

$= i^2(\varepsilon_{kja}\varepsilon_{jeb} - \varepsilon_{kjb}\varepsilon_{jea})$

we use the identity $\boxed{\varepsilon_{iek}\varepsilon_{jmk} = \delta_{ij}\delta_{em} - \delta_{im}\delta_{ej}}$

RHS: $+(\delta_{ka}\delta_{eb} - \delta_{kb}\delta_{ea})$

LHS: $-(-\varepsilon_{kaj}\varepsilon_{ebj} + \varepsilon_{kbj}\varepsilon_{eaj}) =$

$= -(-\delta_{ke}\delta_{ab} + \delta_{kb}\delta_{ae} + \delta_{ke}\delta_{ab} - \delta_{ka}\delta_{be})$ ✓

- the Pauli matrices satisfy

$[\sigma_1, \sigma_2] = 2i\sigma_3$ or $\boxed{[\sigma_j, \sigma_k] = 2i\varepsilon_{jkc}\sigma_e}$

$\Rightarrow$ $\boxed{-\frac{1}{2}\sigma_j \equiv T_j}$ have the same commutation relations as above

$\Rightarrow$ looking at p. 67 we have found a group homomorphism

$$\boxed{\begin{array}{ccc} SO(3) & \longleftrightarrow & SU(2) \\ e^{+iT_a\varphi_a} & \longleftrightarrow & e^{-i\frac{1}{2}\sigma_a\theta_a} \end{array}}$$

by mapping the generators on the level of the algebra

Q: Is this an isomorphism? No

- obviously if in our rep. of $SO(3)$ we rotate by $\varphi = \begin{Bmatrix} 0 \\ 2\pi \end{Bmatrix}$ deg. around the z-axis (or any other axis) we obtain the identity:

$R(0) = R(\varphi = 2\pi) = \mathbb{1}_{3\times 3}$

- lets look at the $SU(2)$ rep. by Pauli matrices

p.67 $A(\theta) = \exp(-\frac{i}{2}\theta_a \sigma_a)$, $\vec{\theta} = \theta \cdot \vec{n}$, choose $\theta = 2\pi$

$\Rightarrow \cos\frac{\theta}{2} = \cos(\pi) = -1$, $\sin\frac{\theta}{2} = \sin\pi = 0$

so $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbb{1}_{2\times 2}$!

- and obviously choosing $\theta = 0$ does give the identity: $A(0) = \mathbb{1}$

$\Rightarrow$ the map $SU(2) \to SO(3)$ has a nontrivial

$\boxed{\text{Kernel } K: \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}}$

- the kernel is a normal subgroup of $SU(2)$ $\cong \mathbb{Z}_2$ (which is also the centre of $SU(2)$, defined in ex) i.e. the set of elements commuting with all group elements)

$\Rightarrow$ $SU(2)/\mathbb{Z}_2$ is a group (p. 35)

and so we have / establish the group isomorphism

$$\boxed{SU(2)/\mathbb{Z}_2 \cong SO(3)}$$