Irreducible reps of SO(3) – an instructive example

- for finite groups we had:

  number of irreps = number of conjugacy classes (finite!)

  $$\sum_{\substack{\mu \\ \text{inequiv} \\ \text{irreps}}} \underbrace{n_\mu^2}_{\text{of dim}} = [g]$$

- for $SO(2) \cong U(1)$ we could label the irreps $e^{-im\varphi}$ by $m \in \mathbb{Z}$

- <u>for SO(3)</u>:

  algebra $[X_a, X_b] = i\,\varepsilon_{abc} X_c \overset{-T_c}{\equiv}$ same as

  $[\hat{L}_a, \hat{L}_b] = i\,\varepsilon_{abc} \hat{L}_c$ $(\hbar = 1)$

  operator $\hat{L} = \hat{x} \times \hat{p}$ in QM, with $[\hat{x}_i, \hat{p}_j] = \delta_{ij}$

$\Rightarrow$ from <u>QM</u>: (irreducible) representations labeled

by $l(l+1)$ eigenvalues of $\hat{L}^2 = \hat{L}_a \hat{L}_a$, with $l \in \mathbb{N}$

$(su(2): l \in \mathbb{N}/2)$

with degeneracy $m = -l, -l+1, \ldots, +l$ $(2l+1)$-fold

for fixed l: eigenvalues of $\hat{L}_3$

<u>basis of irrep $|lm\rangle$</u>:

$$\langle lm' | X_3 | lm \rangle = m\,\delta_{m,m'}$$

$$\langle lm' | X_\pm | lm \rangle = \sqrt{(l \mp m)(l \pm m + 1)}\,\delta_{m', m\pm 1}$$

$X_\pm = X_1 \pm i X_2$

with commutators $[X_3, X_\pm] = \pm X_\pm$

- here the choice of axis $\parallel$ to X_3 is our (arbitrary) convention:

in the basis $|lm\rangle$ we have $X_3 = \begin{pmatrix} l & & & & 0 \\ & l-1 & & & \\ & & \ddots & & \\ & & & -l+1 & \\ 0 & & & & -l \end{pmatrix}$

(for $l = 1$ our T_3 p. 74, after change of basis $X_3 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$)

$$\Rightarrow \quad R_3(\varphi) = e^{-i\varphi X_3} = \mathrm{diag}\left(e^{-il\varphi}, e^{-i(l-1)\varphi}, \dots, e^{+il\varphi}\right)$$

"looks like block-diag with 1d irreps"

with character $\chi^{(l)}(\varphi) = e^{-il\varphi} + e^{-i(l-1)\varphi} + \dots + e^{il\varphi} = e^{-il\varphi}\dfrac{\left(1-e^{(2l+1)i\varphi}\right)}{1-e^{i\varphi}}$

$$= \frac{e^{-i(l+\frac{1}{2})\varphi} - e^{i(l+\frac{1}{2})\varphi}}{e^{-i\varphi/2} - e^{+i\varphi/2}} = \frac{\sin(l+\frac{1}{2})\varphi}{\sin\frac{1}{2}\varphi} = 1 + 2\sum_{m=1}^{l}\cos(m\varphi)$$

* this remains true for rotation by angle φ around an arbitrary axis

Orthogonality of characters of SO(3)

- for finite groups we had $\frac{1}{[g]}\sum_{g\in G} = \frac{1}{[g]}\sum_{i\ \text{conj. classes}} k_i$
- for $SO(2) \cong U(1)$ we replaced this by $\rightarrow \frac{1}{2\pi}\int_0^{2\pi} d\varphi$
- for SO(3): $\rightarrow \int_0^{2\pi} d\mu(\varphi)$

we need to satisfy
- $\int_0^{2\pi} d\mu(\varphi)\cdot 1 = 1$
- in analogy to $\sum_{g\in G} = \sum_{\substack{g'=hg\in G \\ h\in G\ \text{fixed}}}$

the measure has to be invariant under group multiplication

ok: rot by φ or $\varphi+\alpha$

$\equiv$ Haar measure

here: $\boxed{\int d\mu(\varphi) = \int_0^{2\pi} \frac{d\varphi}{2\pi}(1-\cos\varphi)} = \int_0^{2\pi}\frac{d\varphi}{2\pi}\, 2\sin^2\left(\frac{\varphi}{2}\right)$ [Lit: derivation: Appendix C H. F. Jones]

$\cos 2x = \cos^2 x - \sin^2 x$

check: $\delta_{l,l'} \overset{!}{=} \int_0^{2\pi}\frac{d\varphi}{2\pi}\, 2\sin^2\frac{\varphi}{2}\, \frac{\sin\overbrace{(l+\frac{1}{2})\varphi}^{\alpha}}{\sin\frac{\varphi}{2}}\cdot\frac{\sin\overbrace{(l'+\frac{1}{2})\varphi}^{\beta}}{\sin\frac{\varphi}{2}}$, use $\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$ and subtract

$$\overset{!}{=} \int_0^{2\pi}\frac{d\varphi}{2\pi}\left[\cos\left((l+\tfrac{1}{2}-(l'+\tfrac{1}{2}))\varphi\right) - \cos\left((l+\tfrac{1}{2}+l'+\tfrac{1}{2})\varphi\right)\right] \qquad l, l' \in \mathbb{N}$$

$$= \delta_{l,l'} \qquad ♡$$

The Clebsch-Gordon Series – "Addition of Angular Momentum"

- Consider two reps (e.g. 2c p. 60) $D^{(\ell_1)}$, $D^{(\ell_2)}$ with basis $|\ell_{1,2}, m_{1,2}\rangle$ labelled by its eigenvalues ℓ_1 and ℓ_2 respectively.

In QM we learn that the resulting product state has eigenvalues $\ell = |\ell_1 - \ell_2|, \dots, \ell_1 + \ell_2$, or group theoretically

$$D^{(\ell_1)} \otimes D^{(\ell_2)} \overset{!}{=} \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \oplus a_\ell D^{(\ell)}$$

in terms of the irreps $D^{(\ell)}$, with unknown coeffs. How does this come out here? Let $\ell_1 \geq \ell_2$, take the trace:

$$\begin{aligned}
\text{LHS}: \operatorname{Tr}(D^{(\ell_1)} \otimes D^{(\ell_2)}) &= \chi^{(\ell_1)}(\varphi)\, \chi^{(\ell_2)}(\varphi) \\
&= \frac{e^{i(\ell_1+\frac{1}{2})\varphi} - e^{-i(\ell_1+\frac{1}{2})\varphi}}{2i\sin\frac{\varphi}{2}} \cdot \sum_{m_2=-\ell_2}^{+\ell_2} e^{im_2\varphi} \\
&= \frac{1}{2i\sin\frac{\varphi}{2}} \sum_{m=-\ell_2}^{+\ell_2} \left(e^{i(\ell_1+m+\frac{1}{2})\varphi} - e^{-(\ell_1-m+\frac{1}{2})\varphi} \right) \\
&\qquad \text{relabel } m \to -m \\
&= \sum_{m=-\ell_2}^{+\ell_2} \frac{e^{i(\ell_1+m+\frac{1}{2})\varphi} - e^{-(\ell_1+m+\frac{1}{2})\varphi}}{2i\sin\frac{\varphi}{2}} \\
&= \sum_{\ell=-\ell_2+\ell_1}^{\ell_2+\ell_1} \frac{e^{i(\ell+\frac{1}{2})\varphi} - e^{-i(\ell+\frac{1}{2})\varphi}}{2i\sin\frac{\varphi}{2}} = \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \chi^{(\ell)}(\varphi) \\
&\qquad \text{shift } m \to \ell = m + \ell_1 \\
&= \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \oplus \operatorname{Tr} D^{(\ell)} : \text{RHS with } a^{(\ell)} = 1
\end{aligned}$$

Properties of the generators and structure constants

- we have seen that we can parametrise unitary reps of Lie group $G \to GL(n,\mathbb{C})$ by $D(\theta) = \exp\left[i \sum_{a=1}^{\dim[G]} \theta^a T_a\right]$

- the generators T^a form the Lie-algebra $\mathfrak{g}$ of G

- from the closure of multiplication $D(\theta)\, D(\theta')$ and T^a forming a basis we deduced

$$[T_a, T_b] = i f_{ab}{}^c T_c$$ with real structure constants $f_{ab}{}^c = -f_{ba}{}^c \in \mathbb{R}$

- $\mathfrak{g}$ forms a vector space with multiplication $g_1, g_2 \in \mathfrak{g}$

a) $$g_1 \cdot g_2 \equiv [g_1, g_2] = -[g_2, g_1]$$

which is linear $[(\lambda_1 g_1 + \lambda_2 g_2), g_3] = \lambda_1 [g_1, g_3] + \lambda_2 [g_2, g_3]$

b) being defined through the commutators this product satisfies trivially the Jacobi-identity $[[g_1, g_2], g_3] + [[g_3, g_1], g_2] + [[g_2, g_3], g_1] = 0$

$\Rightarrow$ for the generators $g_1 = T_a$, $g_2 = T_b$, $g_3 = T_c$ this implies

$$\boxed{0 = f_{ab}{}^d f_{cd}{}^e + f_{bc}{}^d f_{ad}{}^e + f_{ca}{}^d f_{bd}{}^e}\quad , a, b, \ldots, f \in [1, \ldots, \dim G]$$

[**Note**: one can also define an abstract Lie-product that satisfies (+ commutator) a) and b), e.g. through the Poisson Bracket $[A, B] \equiv \{A, B\}_{PB}$]

• one can also define a scalar product on the vectorspace $\mathfrak{g}$ by $(A,B) \equiv \mathrm{Tr}(A,B)$. The basis T_a $a=1,\dots,\dim[G]$ is called orthogonal if $\mathrm{Tr}(T_a T_b) = T\delta_{ab}$ and orthonormal if $T = \frac{1}{2}$.

* the requirement that the basis T^a is complete leads to a completenessrelation $\sum_{a=1}^{\dim} T^a_{ij} T^a_{kl} = \dots$ (-> exercise for $SU(N)$)

Representations and Lie-Algebras:

• we can also consider more general reps $D: G \to V$ for V some vectorspace of dim d, that are not necessarily unitary. What we've said remains true: $D = \exp[i \sum_{a=1}^{\dim} \theta^a T^a]$, with the same # of T^a's but these are no longer hermitian in general, and are of dim $d \times d$.

* we distinguish 3 important examples:

the conjugate rep $*$: $D_* = \exp[-i \sum_{a=1}^{\dim} \theta^a T_a^*]$

=> the generators $-T_a^*$ satisfy the same Lie-algebra (which is rep. indep.)

$$[T_a, T_b] = i f_{ab}{}^c T_c \Leftrightarrow [-T_a^*, -T_b^*] = -i f_{ab}{}^c T_c^*$$

the adjoint rep (A)

the generators $(F_a)_b{}^c = -i f_{ab}{}^c$ satisfy the same Lie-algebra (ex 8.2)

Because $b, c = 1, \dots, \dim[G]$ this is the dimension $d = \dim[G]$ of the rep.
e.g. $SU(n) \Rightarrow d = n^2 - 1$

the fundamental rep (F) as before which is $d = n$ dimensional for $(S)U(n)$, $(S)O(n)$. It is often denoted by $\mathbf{n}$.