Some reminders & formal definition of a Lie Algebra

- G Lie group, parametrised by $\dim[G]$ real parameters

$\Rightarrow$ $D(G)\begin{cases} g \to \exp\left[i \sum_{a=1}^{\dim[G]} \theta^a T^a\right] \\ G \to GL(n, \mathbb{C}) \end{cases}$ is an $n\times n$ dimensional unitary matrix rep of G

which lies in the component K of G containing the identity.

The generators T^a $a=1..,\dim[G]$ satisfy $T^a = T^{a\dagger}$,

and if furthermore $\det D(\theta) = 1$ $\forall \theta$ $\Rightarrow$ $\mathrm{Tr}\, T^a = 0$ $\forall a$

Furthermore $[T^a, T^b] = i f^{ab}{}_c T^c$ holds, with $f^{ab}{}_c \in \mathbb{R}$.

* on p. 38 we defined a vectorspace over a field

- The $n\times n$ dim generators span a $\dim[G]$ dimensional vector space over $\mathbb{C}$, with elements

  $\theta = \theta^a T^a$, addition $\theta + \varphi = \theta^a T^a + \varphi^a T^a = (\theta^a + \varphi^a) T^a$

  and scalar multiplication $\alpha\theta = (\alpha\theta^a)T^a$, satisfying all rules A0-A4 and B0-B4 on p 38

- Def A vector space V over field K is called an Algebra over K if there exists a binary operation (multiplication) $V\times V \to V$ that satisfies

$\forall x, y, z \in V$ $\quad (x+y)\cdot z = x\cdot z + y\cdot z$ $\quad$ left & right
$\quad\quad\quad\quad\quad\quad x\cdot(y+z) = x\cdot y + x\cdot z$ $\quad$ distributivity

$\forall \alpha \in K$ $\quad \alpha(x\cdot y) = (\alpha x)\cdot y = x\cdot(\alpha y)$

Def a vector space V over K is called a Lie Algebra

if the binary operation $X \cdot Y \equiv [X, Y]_L$

satisfies in addition $V \times V \longrightarrow V$

$[X, X]_L = 0$ and the Jacobi identity holds,

$$[X, [Y, Z]_L]_L + [Z, [X, Y]_L]_L + [Y, [Z, X]_L]_L = 0$$

Such a binary op. is then called a Lie bracket.

Examples for Lie bracket for matrices are the commutator or the Poisson-bracket

$\Rightarrow$ equipped with the binary operation given by the commutator of the $n \times n$ matrices

$$T^a \cdot T^b \equiv [T^a, T^b] = i f^{ab}{}_c T^c$$

$$g \times g \longrightarrow g$$

the vector space g of our generators T^a, $a = 1..\dim[G]$

is a Lie-algebra in the above sense

• one can also define a scalar product on the vector space $\mathfrak{g}$ by $(A,B) = Tr(AB)$. The basis T_a $a=1,\dots,\dim[G]$ is called orthogonal if $Tr(T_a T_b) = t_a \delta_{ab}$ (no sum) and orthonormal if $T = \frac{1}{2}$.

* the requirement that the basis T^a is complete leads to a completeness relation $\sum_{a=1}^{\dim} T^a_{ij} T^a_{ke} = \dots$ (-> exercise for $SU(n)$) idea see next page

Representations and Lie-Algebras:

• we can also consider more general reps $D: G \to V$ for V some vectorspace of dim d, that are not necessarily unitary. What we've said remains true: $D = \exp[i \sum_{a=1}^{\dim} \theta^a T^a]$, with the same # of T^a's but these are no longer hermitian in general, and are of dim $d \times d$.

* we distinguish 3 important examples:

the conjugate rep $*$: $D_* = \exp[-i \sum_{a=1}^{\dim} \theta^a T^*_a]$

$\Rightarrow$ the generators $-T^*_a$ satisfy the same Lie algebra (which is rep. indep.)

$$[T_a, T_b] = i f_{ab}{}^c T_c \Leftrightarrow [-T^*_a, -T^*_b] = -i f_{ab}{}^c T^*_c$$

the adjoint rep (A)

the generators $(F_a)_b{}^c = -i f_{ab}{}^c$ satisfy the same Lie-algebra (ex 8.2)

Because $b,c = 1,\dots,\dim[G]$ this is the dimension $d = \dim[G]$ of the rep,
e.g. $SU(n) \Rightarrow d = n^2 - 1$

the fundamental rep (F) as before which is $d = n$ dimensional for $(S)U(n)$, $(S)O(n)$. It is often denoted by $\mathbf{n}$.

$M = \theta_a T^a$ choose $Tr(T_a T_b) = T \delta_{ab}$ ON basis

$\Rightarrow Tr(M T^b) = Tr(\theta_a T^a T^b) = \theta_a \delta^{ab} T = \theta^b T$

$\Rightarrow M = \frac{1}{T}\left(Tr(M T^a)\right) T^a$

$M_{ij} = \frac{1}{T} \quad M_{kl} T^a_{lk} \; T^a_{ij}$

$\Rightarrow$

- these reps differ
  - in the normalisation of the scalar product T
  - in the dimension d
  - in the quadratic Casimir operator C:

$$\boxed{\sum_{a=1}^{\dim} (T_a T_a)_{ij} = C\, \delta_{ij}}$$

eg. SU(2): - fund. F has $T_F = \frac{1}{2}$ & orthonormal ($= t_a t_a$), $\operatorname{Tr}(T_a^F T_b^F) = \delta_{ab} \cdot \frac{1}{2}$, $T_a = -\frac{1}{2}\sigma_a$ | SU(n): $d_F = n$, $C_F = \frac{n^2-1}{n}$ (from completeness relation)

SU(n) - adj. A has $T_A = n$ ($= t_a t_a$) & orthogonal, $\operatorname{Tr}(T_a^A T_b^A) = n\,\delta_{ab}$, $d_A = n^2-1$ ($= \dim[G]$), $C_A = n$

* from the adjoint rep we can learn more about the $f_{ab}{}^c$:

The corresp. scalar prod. is called, $A, B \in G$

killing form $\boxed{(A, B) \equiv \operatorname{Tr}(D_A(A)\, D_A(B))}$ adj rep

applied to the generators it is called

Cartan metric $\boxed{g_{ab} \equiv \operatorname{Tr}_A(T_a^A T_b^A) =} - f_{ac}{}^d f_{bd}{}^c = g_{ba}$ (Tr cyclic)

let us define $f_{abc} \equiv f_{ab}{}^d g_{dc}$

$\Rightarrow$ f_{abc} is totally antisym. in all indices (we were sloppy in notation before, but due to $g_{ab} \sim \delta_{ab}$ for su(n) this didn't matter there)

proof: $\operatorname{Tr}([A,B]C) = \operatorname{Tr}(ABC - BAC)$
$= \operatorname{Tr}([B,C]A) = \operatorname{Tr}([C,A]B)$ due to cyclicity

now $\operatorname{Tr}_A([T_a^A, T_b^A]\, T_c^A) = i f_{ab}{}^d \underbrace{\operatorname{Tr}_A(T_d^A T_c^A)}_{g_{dc}} = i f_{abc} = i f_{cab} = -i f_{acb}$ etc