Subalgebras, Ideal and (semi-) simple Lie Algebras

- the notion of a subgroup or of a normal (=invariant) subgroup of a Lie-group G have their counter parts on the Lie-algebra $\mathfrak{g}$, they will be useful in the classification of Lie-algebras.

* if we have a subgroup $H \subseteq G$ (that forms a group itself) we also must have a

Def subalgebra : given a Lie-algebra $\mathfrak{g}$ with generators T_a, $a = 1, \dots, r$. A subalgebra $\mathfrak{g}'$ is a subset $\mathfrak{g}' = \{S_a = T_a \text{ for } a = 1, \dots, s \leq r\} \subseteq \mathfrak{g}$ that closes under multiplication $[S_a, S_b] = i f_{ab}{}^c S_c$ $\quad a,b,c = 1, \dots, s$ and thus forms an algebra itself.

- Note that a subalgebra $\mathfrak{g}'$ spans a subgroup $H \subseteq G$ by def. $h = \exp[i\theta^a S_a] \Leftrightarrow h \in H$.

- We can spot the existance of a subalgebra iff there exists an s, $1 \leq s \leq r$:
$$f_{ab}{}^c = 0 \quad \forall a,b = 1, \dots, s \quad \forall c = s+1, \dots, r$$

- When the Lie-group G is Abelian we must have
$$\Longleftrightarrow \quad [T_a, T_b] = 0 \quad \forall a,b = 1, \dots, \dim[G]$$
(proof: Baker-Campbell-Hausdorff formula $e^A e^B e^{\frac{1}{2}[B,A]} = e^{A+B} = e^{B+A}$)

likewise we have an Abelian sub-group $^{\dim H = s}$ when the corresp. subalgebra is commutative : $[S_a, S_b] = 0 \quad \forall a,b = 1, \dots, s \leq r$

- we again speak of a proper subalgebra if its dim s has $1 < s < \dim G$

- recall that a subgroup H of group G, $H \subseteq G$ was called normal (= invariant) subgroup if it holds:

$$\forall g \in G \quad gH = Hg \iff gHg^{-1} = H$$

Such a normal subgroup H is generated by

Def An ideal I or invariant subalgebra of Lie-algebra $\mathfrak{g}$ is a subalgebra ($\mathfrak{g}^{\dim}_{s}$) that satisfies $\forall S_a \in I \; \forall T_b \in \mathfrak{g}$ $\boxed{[S_a, T_b] = i f_{ab}{}^c S_c}$ "$[I, \mathfrak{g}] = I$"

or in other words $f_{ab}{}^c = 0 \;\; \forall a = 1,\dots,s \;\; \forall c = s+1,\dots,r$

Note: that here the index b is not constrained, in contrast to a subalgebra

* To see the relation we go back to p. 71, 72 where we multiplied infinitesimal group elements. For $gH = Hg$ $\forall g \in G$ we must find $\forall h \in H \; \forall g \in G \; \exists h' \in H$ s.th. $gh = h'g$:

$$[\mathbb{1} + i\theta^a T_a - \tfrac{1}{2}\theta^a\theta^b T_a T_b][\mathbb{1} + i\varphi^a S_a - \tfrac{1}{2}\varphi^a\varphi^b S_a S_b]$$

$\stackrel{!}{=}$

with $\varphi^a \to \varphi'^a$

this is possible iff $[S_a, T_b] = i f_{ab}{}^c S_c \;\; \forall a = 1\dots s \; \forall b = 1,\dots,r$

and we have $\varphi'^c = \varphi^c - f_{ab}{}^c \theta^a \varphi^b$

Def A Lie-algebra $\mathfrak{g}$ is called simple if it does not contain any proper ideal (only: no generators ($\mathbb{1}$) or the entire $\mathfrak{g}$)

Note: a single generator T_a is always a subalgebra, but not necessarily invariant.

Def A Lie-algebra is called semi-simple if it does not contain any proper Abelian ideal.

- recall the definition of G being the direct product (p. 31A) of its subgroups A and B : $G = A \times B$ ($\neq$ direct product reps (*))
iff i) all elements of A commute with all elements of B
ii) all elements g of G can be written in a unique way as $g = a \cdot b$ with $a \in A$, $b \in B$

This induces the following structure on the Lie-algebra:

Def: the Lie-algebra $\mathfrak{g}$ is the direct sum of its subalgebras $\mathfrak{g} = g_1 \oplus g_2$ (with $\mathfrak{g} = g_1 \cup g_2$) iff the corresp. Lie-group $G = G_1 \times G_2$ is the direct product of the corresp. subgroups $G_{1,2}$ spanned by $g_{1,2}$.

The Cartan metric provides a necessary and sufficient criterium if Lie-algebra $\mathfrak{g}$ is semi-simple: (without proof)

$$\Leftrightarrow \boxed{\begin{array}{l} \det g_{ab} = \det\left[\mathrm{Tr}_{A}(T_a T_b)\right] \neq 0 \\ \Leftrightarrow \text{the killing form is non-degenerate:} \\ \forall X \in \mathfrak{g} \quad (A, X) = \mathrm{Tr}(D_A(A) D_A(X)) = 0 \Rightarrow A = 0 \end{array}}$$

*An important result for the classification is:

Prop: Any semi-simple Lie-algebra $\mathfrak{g}$ can be written as a direct sum of simple Lie-algebras:

proof: If g is already simple we are done.
Suppose $\mathfrak{g}$ has an ideal I (non-Abelian!). Define by P its orthogonal complement w.r.t. the Killing form:
$$P \in P \Leftrightarrow (S_a, P) = 0 \quad \forall S_a \in I.$$

P itself is a subalgebra, that is the product of $\rho_1, \rho_2 \in P$

$$[\rho_1, \rho_2] \in P:$$

$$\forall S^a \in I: ([\rho_1, \rho_2], S_a) \underset{\text{Tr cyclic}}{\overset{!}{=}} (\underbrace{[\rho_2, S_a]}_{\text{in } I \text{ as idea}}, \rho_1) = f_{2a}{}^{c}(\underbrace{S_c, \rho_1}_{0}) = 0$$

* To show that $g = I \oplus P$ we still need to show that all elements from I and P commute:

Consider $S_a \in I$, $\rho_b \in P$: $[\rho_b, S_a] \overset{!}{=} 0$:

- we have $([\rho_b, S_a], T_d) \underset{\text{cyclic}}{=} (\underbrace{[S_a, T_d]}_{i f_{ad}{}^{c} S_c \text{ as } I \text{ idea}}, \rho_b) = i f_{ad}{}^{c}(S_c, \rho_b) = 0$

so $[\rho_b, S_a]$ is orthogonal to any element in g. Because the killing form is non degenerate (Ideal non-Abelian) $[\rho_b, S_a] = 0 \quad \forall a, b$

- because any $g \in g$ is either in I or in P (generators T^a form the basis of a vector space) we have

$$g = I \oplus P.$$

In case P is simple we are done, otherwise we continue. Because the number of generators is finite $= \dim [G]$ to begin with, eventually $g = \sum_i \oplus g_i$ with g_i simple.

• **Note**: to any Lie-Algebra ($\equiv$ satisfying a) & b) on p. 80)

there exists a corresponding Lie-group. (Recall we only proved the rep. theorem D = explicitly partly)

Thus classifying all Lie-algebras amounts to finding all real solutions to the Jacobi-identity b) p. 80 for a given index range $a = 1, \ldots, n \in \mathbb{N}$. Will only cover part of this program, for compact semi-simple Lie-algebras:

Def: $\mathfrak{g}$ is compact if it is a real Lie-alg. and it's Cartan metric g_{ab} positive definite (see p. 82)

• as a final remark:

- the structure constants specify the Lie-alg.
- the Lie-alg. doesn't necessarily specify the $f_{ab}{}^c$, as in any vector space we may change basis $\Rightarrow f_{ab}'{}^c$

The Cartan Basis of a Lie Algebra

In order to illustrate its concept let us recall the algebra $su(2)$: $\boxed{[J_a, J_b] = i\varepsilon_{abc} J_c}$

by building the linear combinations $J_\pm = J_1 \pm i J_2$

which is completely equiv.

we obtain

$$\boxed{\begin{aligned} [J_3, J_\pm] &= \pm J_\pm \\ [J_+, J_-] &= 2 J_3 \end{aligned}}$$

For a general Lie algebra the Cartan basis divides it into a set of commuting operators (here only 1: J_3) and a set of lowering and raising operators, here $J_\pm$ (also called step op.)