Def The Cartan subalgebra is the maximal set of commuting generators $\{H_i\}$ $i = 1, \ldots, r$. The number r is the rank of the algebra (or group): $[H_i, H_j] = 0 \quad \forall i,j = 1, \ldots, r$. $(\leq \dim[G])$

example: $SU(n)$ has rank $n-1$, because we can have at most $n-1$ independent diagonal $n \times n$ matrices ($\Rightarrow$ they commute) with real entries (its generators are Hermitian) that are traceless ($\exists$ n diag matrices $\begin{pmatrix} 1 & & \\ & 0 & \\ & & \ddots \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 1 & \\ & & \ddots \end{pmatrix}, \ldots, \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$ minus the condition $tr = 0$).

$\Rightarrow$ $SU(2)$ has rank $n-1 = 1$ $\quad H_1 = J_3$

$SU(3)$ " " $n-1 = 2$

example: the generators of $SU(3)$: Gell-Mann matrices

generators $\boxed{T_a = \frac{1}{2}\lambda_a}$, $a = 1, 2, \ldots, 8 = n^2 - 1$

with

$\lambda_a = \begin{pmatrix} \sigma_a & 0 \\ & 0 \\ 0 \; 0 & 0 \end{pmatrix}$ $\quad a = 1, 2, 3$ $\quad SU(2)$ subalgebra

$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$

for ON scalar prod. p. 81

$\lambda_6 = \begin{pmatrix} 0 & 0 \; 0 \\ 0 & \sigma_1 \\ 0 & \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 \; 0 \\ 0 & \sigma_2 \\ 0 & \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

the Cartan subalgebra is formed by T_3 and T_8 $= \{H_1, H_2\}$

- from Linear Algebra (& QM) we know that commuting matrices (operators) can be simultaneously diagonalized, with their eigenvalues characterising the rep.

• for a Lie-algebra of dimension d we have $d-r$ remaining generators,

Aim: form linear combinations E_α, $\alpha = 1, \dots, d-r$
s.th. they behave like step operators w.r.t. all $H_{i=1,\dots,r}$:

$$[H_i, E_\alpha] \sim E_\alpha \qquad \text{just as in } su(2) \left(\begin{array}{l} H_1 = J_3 \\ E_{1,2} = J_\pm \end{array} \right)$$

• this is equivalent to saying that for all generators

$X \in \{H_1, \dots, H_r, E_1, \dots, E_{d-r}\}$ $\quad [H_i, X] \sim X$, where the coeff. on the RHS is zero if $X = H_j$.

* $[H_i, X]$ is called the adjoint action of H on X, which we wish to diagonalize

→ we are looking at the solution of an eigenvalue eq.

$$[H_j, E_\alpha] = i f_{j\alpha}{}^{\beta} E_\beta = \lambda E_\alpha \qquad \text{for a given } j$$

$$\Leftrightarrow \quad i f_{j\alpha\beta} E^\beta = \lambda g_{\alpha\beta} E^\beta$$

$\Rightarrow$ solving $\quad \boxed{\det(i f_{j\alpha\beta} - \lambda g_{\alpha\beta}) = 0} \quad$ for j fixed,

with $f_{j\alpha\beta} = -f_{j\beta\alpha}$ real $i f_{j\alpha\beta}$ is Hermitian and has real eigenvalues:
we already know that r of them are zero.

Theorem (Cartan): The eigenvalues $(\alpha)_j \neq 0$, $\alpha = 1 \dots d-r$, j fixed
in $[H_j, E_\alpha] = (\alpha)_j E_\alpha$ are non-degenerate. $\quad (\alpha_j = 0 \to E_\alpha \to H)$

(that is there is only 1 eigenvector with that eigenvalue).

* what about the remaining eigenvectors?

$\Rightarrow$ Consider $[H_k, E_\alpha]$ for $k \neq j$: we have

$$[H_j,[H_k,E_\alpha]] \overset{\text{Jacobi}}{=} [H_k,[H_j,E_\alpha]] - [E_\alpha, \underbrace{[H_j, H_k]}_{=0}]$$

$$= (\alpha)_j \, [H_k, E_\alpha]$$

$\Rightarrow$ $[H_k, E_\alpha]$ is also an eigenvector of H_j, with the same eigenvalue

$\Rightarrow$ the non-deg. of eigenvalues implies that we must have

$$\boxed{[H_k, E_\alpha] \sim E_\alpha}$$

and thus we have $\boxed{= (\alpha)_k E_\alpha}$ $\forall k = 1, .., \cancel{j}, .., r$ too

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<u>Def</u>: The r-dim vectors

$$\left.\begin{array}{l} \alpha \equiv ((\alpha)_1, .., (\alpha)_r) \\ \beta \equiv ((\beta)_1, .., (\beta)_r) \\ \vdots \end{array}\right\} d-r$$

are called the <u>roots</u> α of the Lie-algebra,

their <u>eigenvectors E_α</u> are called <u>root vectors</u> (or step op.)

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* Commutation relations among root vectors?

$$[E_\alpha, E_\beta] = ?$$

Consider $[H_j, [E_\alpha, E_\beta]] \overset{\text{Jacobi}}{=} -[E_\alpha,[E_\beta,H_j]] - [E_\beta[H_j,E_\alpha]]$

$$= (\alpha+\beta)_j \, [E_\alpha, E_\beta]$$

$\Rightarrow$ $[E_\alpha, E_\beta]$ is $\begin{cases} \text{also a root vector with root } \alpha+\beta \\ \text{or } \alpha+\beta = 0 \\ \text{or } [E_\alpha, E_\beta] = 0 \end{cases}$

- for $\alpha+\beta=0$ (or $(\alpha+\beta)_j=0$ $\forall j$ in components)

$[E_\alpha, E_{-\alpha}]$ commutes with all H_j and so is element of the Cartan subalgebra (linear comb thereof = sum over $j=1,2,\dots,r$)

$$\boxed{\begin{aligned}[E_\alpha, E_{-\alpha}] &= \lambda_j H_j \\ [E_\alpha, E_\beta] &= N_{\alpha\beta} E_{\alpha+\beta}\end{aligned}}$$

else $\qquad (\beta \neq -\alpha)$

with $N_{\alpha\beta}=0$ where $\alpha+\beta$ is not a root

- Why is with α also $\beta=-\alpha$ a root?

Take the transposed of $\det(i f_{j\alpha\beta} - \lambda g_{\alpha\beta})^T = \det(-i f_{j\alpha\beta} - \lambda g_{\alpha\beta}) = 0$

f antisym, g sym. $\Rightarrow -\lambda$ ev or $\lambda = 0$

- in the particular case of $H_j = H_j^+$ (as is true for $su(n)$)

we have $[H_j, E_\alpha]^+ = (\alpha)_j^* E_\alpha^+ = -[H_j, E_\alpha^+] = (\alpha)_j E_\alpha^+$

$(\alpha)_j \in \mathbb{R}$

so $E_{-\alpha} = E_\alpha^+$.

Scalar products in the Cartan basis:

- $Tr_A([H_j, E_\alpha] H_k) \overset{Tr\ cyclic}{=} -Tr_A([H_j, H_k] E_\alpha) = 0$

$= Tr_A((\alpha)_j E_\alpha H_k) = \underset{\neq 0}{(\alpha)_j} (E_\alpha, H_k) \Rightarrow \boxed{(E_\alpha, H_k) = 0}$

- $Tr_A([H_j, E_\alpha] E_\beta) = Tr_A(E_\alpha [E_\beta, H_j]) = -(\beta)_j Tr_A(E_\alpha E_\beta)$

$= (\alpha)_j Tr_A(E_\alpha E_\beta) \Leftrightarrow 0 = (\alpha+\beta)_j Tr_A(E_\alpha E_\beta) = (\alpha+\beta)_j (E_\alpha, E_\beta)$

$\Rightarrow$ when $\alpha+\beta \neq 0$ $\boxed{(E_\alpha, E_\beta) = 0}$, i.e. $\alpha \neq 0 \Rightarrow (E_\alpha, E_\alpha) = 0$

* For $\mathfrak{g}$ semi-simple : $\det g_{\alpha\beta} \neq 0$

with $g_{\alpha\beta} = \mathrm{tr}_{\mathrm{ad}}(T_\alpha T_\beta) = (T_\alpha, T_\beta)$

- if we label our E's as $\underbrace{E_\alpha, E_{-\alpha}, E_\beta, E_{-\beta}, \dots, E_\gamma, E_{-\gamma}}$

we already have for the scalar-product $d-r$

$$
\begin{array}{c|c|cc}
g_{\alpha\beta} & H_1 \dots H_r & E_\alpha \; E_{-\alpha} & \dots \; E_{-\gamma}, E_\gamma \\
\hline
H_1 \;\vdots\; H_r & r \times r & \multicolumn{2}{c}{0} \\
\hline
E_\alpha & & \begin{array}{cc} 0 & \\ & 0 \end{array} & 0 \\
E_{-\alpha} & 0 & \multicolumn{2}{c}{(d-r)\times(d-r)} \\
\vdots & & & \\
E_{-\gamma} & & 0 & \\
(E_\gamma) & & &
\end{array}
$$

$\Rightarrow$ in order to have $\det g_{\alpha\beta} \neq 0$ we need

- $(E_\alpha, E_{-\alpha}) \neq 0$, normalise s.th. $(E'_\alpha, E'_{-\alpha}) = 1$
- the commuting sector = Cartan subalgebra is a real sym. matrix (H_i, H_j) $\Rightarrow$ diagonalise and choose $(H'_i, H'_j) = \delta_{ij}$ because the H_i all commute

new linear combinations also commute $H_i \to H'_i$

- after choosing this normalisation we can determine the λ_i in

$$[E'_\alpha, E'_{-\alpha}] = \lambda_i H'_i \overset{!}{=} (\alpha)_i H'_i$$

we have (dropping ')

$([E_\alpha, E_{-\alpha}], H_i) = (\lambda_j H_j, H_i) = \lambda_j \delta_{ji} = \lambda_i$ and also

$\underset{\text{cyclic}}{=} ([H_i, E_\alpha], E_{-\alpha}) = ((\alpha)_i E_\alpha, E_{-\alpha}) = (\alpha)_i \cdot 1$ ✓

$\Rightarrow$ in the $\boxed{\text{Cartan - Weyl basis}}$ we have f. $\underline{\text{of semi-simple}}$:

commutators	scalar products
$[H_i, H_j] = 0$	$(H_i, H_j) = \delta_{ij}$
$[H_i, E_\alpha] = (\alpha)_i E_\alpha$	$(H_i, E_\alpha) = 0$
$[E_\alpha, E_{-\alpha}] = (\alpha)_i H_i$	$(E_\alpha, E_{-\alpha}) = 1$
$[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}$ for $\alpha \neq -\beta$	$(E_\alpha, E_\beta) = 0 \quad \alpha \neq -\beta$

"Quantisation" of the Root Vectors:

* we will show now that the root vectors can only take certain length and certain angles

- scalar products of root vectors and of the Cartan subalgebra

  we have $\alpha = ((\alpha)_1, \dots, (\alpha)_r)$ is r-dim

  $H_1, \dots, H_r$ are r generators

$\Rightarrow$ def $\alpha \cdot \beta \equiv (\alpha)_j (\beta)_j$ (summation over $j = 1..r$)

$\alpha^2 \equiv \alpha \cdot \alpha, \quad |\alpha| \equiv \sqrt{\alpha^2}$ ($\neq$ as ev $(\alpha)_i \neq 0$)

Def: $\boxed{H_\alpha \equiv \frac{2}{\alpha^2} \alpha \cdot H}$ $\exists$ max r linearly indep H_α's

$[H_i, H_j] = 0 \Rightarrow \underline{[H_\alpha, H_\beta] = 0}$