we have (dropping ')

$$([E_\alpha, E_{-\alpha}], H_i) = (\lambda_j H_j, H_i) = \lambda_j \delta_{ij} = \lambda_i \text{ and also}$$

$$\underset{\text{cyclic}}{=} ([H_i, E_\alpha], E_{-\alpha}) = ((\alpha)_i E_\alpha, E_{-\alpha}) = (\alpha)_i \cdot 1 \quad ✓$$

$\Rightarrow$ in the $\boxed{\text{Cartan-Weyl basis}}$ we have f. of semi-simple:

| commutators | scalar products |
|---|---|
| $[H_i, H_j] = 0$ | $(H_i, H_j) = \delta_{ij}$ |
| $[H_i, E_\alpha] = (\alpha)_i E_\alpha$ | $(H_i, E_\alpha) = 0$ |
| $[E_\alpha, E_{-\alpha}] = (\alpha)_i H_i$ | $(E_\alpha, E_{-\alpha}) = 1$ |
| $[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}$ for $\alpha \neq -\beta$ | $(E_\alpha, E_\beta) = 0 \quad \alpha \neq -\beta$ |

"Quantisation" of the Root Vectors:

* we will show now that the root vectors can only take certain length and certain angles

- scalar products of root vectors and of the Cartan subalgebra

we have $\alpha = ((\alpha)_1, \ldots, (\alpha)_r)$ is r-dim

$H_1, \ldots, H_r$ are r generators

$\Rightarrow$ def

$$\alpha \cdot \beta \equiv (\alpha)_j (\beta)_j = \beta \cdot \alpha \quad (\text{summation over } j = 1 \ldots r)$$

$$\alpha^2 \equiv \alpha \cdot \alpha, \quad |\alpha| \equiv \sqrt{\alpha^2} \quad (\neq \text{as ev } (\alpha)_i \neq 0)$$

Def: $\boxed{H_\alpha \equiv \frac{2}{\alpha^2} \alpha \cdot H}$ $\exists$ max r linearly indep $H_\alpha$'s

$$[H_i, H_j] = 0 \Rightarrow [H_\alpha, H_\beta] = 0$$

$$2\frac{\alpha_i}{\alpha^2}[H_i, E_\beta] = 2\frac{(\alpha)_i(\beta)_i}{\alpha^2} E_\beta$$ eigenvalue eq

$$\Leftrightarrow \boxed{[H_\alpha, E_\beta] = \frac{2\alpha\cdot\beta}{\alpha^2} E_\beta}$$ , in particular $\boxed{[H_\alpha, E_{\pm\alpha}] = \pm 2 E_{\pm\alpha}}$

and $$\boxed{[E_\alpha, E_{-\alpha}] = \frac{\alpha^2}{2} H_\alpha}$$ with $J_3 = \frac{1}{2} H_\alpha$, $J_\pm = \sqrt{\frac{2}{\alpha^2}}\, E_{\pm\alpha}$

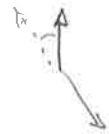
this is an $\underline{su(2)}$ (p. 87)

subalgebra

- for each $\alpha$ $E_{\pm\alpha}$ act as step operators wrt $H_\alpha$
- QM: the eigenvalues of $J_3$ are halfinteger $\Rightarrow$ of $\boxed{H_\alpha \text{ integer}}$

$\Rightarrow$ (1) $$\boxed{\frac{2\alpha\cdot\beta}{\alpha^2} = n \in \mathbb{N}}$$ $n = 0, 1, 2, \ldots$

(2) Cauchy-Schwarz $\alpha\cdot\beta \leq |\alpha||\beta|$

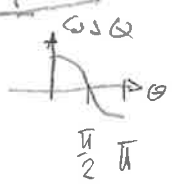


$$\Leftrightarrow \boxed{\underbrace{\left(\frac{2\alpha\cdot\beta}{\alpha^2}\right)}_{n_1} \cdot \underbrace{\left(\frac{2\beta\cdot\alpha}{\beta^2}\right)}_{n_2} \leq 4}$$

$$n_1 \cdot n_2 \leq 4$$

- for the $\underline{\text{angle between } \alpha \text{ and } \beta}$ we have $\boxed{\cos\theta = \frac{\alpha\cdot\beta}{|\alpha||\beta|}}$

$\to$ we may choose $\underline{\cos\theta = +\frac{1}{2}(n_1 n_2)^{\frac{1}{2}}}$ with $\theta \in [0, \frac{\pi}{2}]$




for an appropriate choice of $\pm\alpha, \pm\beta$, the remaining possibilities are reached by $\alpha \to -\alpha$, $\beta \to -\beta$, $\alpha \leftrightarrow \beta$

- without loss of generality we may choose

$$\boxed{0 \leq n_2 \leq n_1} \leq 4$$

• the case $\underline{n_2 = 1 \quad n_1 = 4}$ would saturate Cauchy Schwarz
$(\cos\theta = 1, \theta = 0 \quad \alpha \parallel \beta)$

can hence lead to $\underline{\beta = 2\alpha}$ ; $n_1 = \frac{2\alpha\cdot(2\alpha)}{\alpha^2} = 4, \; n_2 = \frac{2(2\alpha)\cdot\alpha}{2^2\alpha^2} = 1$

This can be excluded as follows:

– assume $\beta = 2\alpha$ is a root, as well as is $\alpha$

* $2\alpha$ cannot be generated from $E_\alpha$ as $[E_\alpha, E_\alpha] = 0$ trivially

* $3\alpha$ cannot be a root as well $\left( \Rightarrow n_1 = \frac{2\alpha\cdot(3\alpha)}{\alpha^2} = 6 > 4, \; n_2 = \frac{2(3\alpha)\cdot\alpha}{9\alpha^2} \notin \mathbb{N} \; ↯ \right)$

$\Rightarrow \underline{[E_\alpha, E_{2\alpha}] = 0}$

* consider $[E_{-\alpha}, E_{2\alpha}] = ?$

$\underline{= 0}$ is impossible: $[[E_{-\alpha}, E_\alpha], E_{2\alpha}] = [-\frac{1}{2}\alpha^2 H_\alpha, E_{2\alpha}]$

$= -\frac{1}{2}\alpha^2 \frac{2\alpha\cdot 2\alpha}{\alpha^2} E_{2\alpha} = -\alpha^2 E_{2\alpha}$

on the other hand Jacobi implies

$= -[\underbrace{[E_\alpha, E_{2\alpha}]}_{=0 \text{ above}} E_{-\alpha}] - [[E_{2\alpha}, E_{-\alpha}], E_\alpha] \overset{!!!}{=} 0 \quad \Rightarrow E_{2\alpha} \equiv 0 \; ↯$

$\underline{\text{else}\neq 0}$: $[E_{-\alpha}, E_{2\alpha}] = N_{-\alpha\, 2\alpha}^{\neq 0} E_\alpha$ :

but $[\underbrace{[E_{-\alpha}, E_{2\alpha}]}, E_\alpha] = -[[E_{2\alpha}, E_\alpha], E_{-\alpha}] - [[E_\alpha, E_{-\alpha}], E_{2\alpha}]$

$0 = [N_{-\alpha, 2\alpha} E_\alpha, E_\alpha] \qquad \overset{\shortparallel}{0} \qquad [\frac{1}{2}\alpha^2 H_\alpha, E_{2\alpha}] = \alpha^2 E_{2\alpha}$

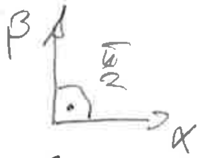
$↯$

* from all possible integers we may also exclude

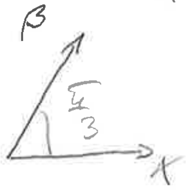
$\underline{n_1 = n_2 = 2}$ as this is $\underline{\text{the trivial case } \alpha = \beta}$

* $\exists$ 4 possible integer pairs. $0 \le n_2 \le n_1$

a) $\underline{n_1 = 0 = n_2}$ $\quad \theta = \frac{\pi}{2}$ $\quad$ no relation between $|\alpha|$ and $|\beta|$




b) $\underline{n_1 = 1 = n_2}$ $\quad \theta = \frac{\pi}{3}$ $\quad |\alpha| = |\beta|$




c) $\underline{n_1 = 2,\ n_2 = 1}$ $\quad \theta = \frac{\pi}{4}$ $\quad |\beta| = \sqrt{2}\,|\alpha|$


d) $\underline{n_1 = 3,\ n_2 = 1}$ $\quad \theta = \frac{\pi}{6}$ $\quad |\beta| = \sqrt{3}\,|\alpha|$


* For $\underline{\text{rank } r = 2}$ $\quad \alpha = ((\alpha)_1, (\alpha)_2)$ $\quad \beta = ((\beta)_1, (\beta)_2)$ are vectors in 2d,

and so we only have the 4 possibilities indicated.

* The relation between <u>any</u> 2 roots must satisfy the same constraints

<u>Note</u>: we also have the roots $-\alpha, -\beta$ then, as well as the Weyl reflections ($\to$ exercise)

Q: How can we <u>generate all roots</u>?

. <u>repeated commutations of $E_\alpha$ with $E_{\pm\beta}$ generates</u>

$[E_\alpha, E_{\pm\beta}] = N_{\alpha,\pm\beta}\, E_{\alpha\pm\beta}$ etc. so the following roots forming

<u>the $\beta$-string</u>: $\alpha + p\beta, \ldots, \alpha - q\beta$ $\qquad p, q \geq 0$ integer

. they form an "irrep. of $\frac{1}{2}H_\beta$" of dim $2j+1$, $j \in \frac{\mathbb{N}}{2}$ or $\in \mathbb{N}$:

at the upper and lower end the $(J_3)$ eigenvalue must be $+j$ and $-j$ resp.

$$\underline{j}\, E_{\alpha+p\beta} = [\tfrac{1}{2}H_\beta, E_{\alpha+p\beta}] = \underline{\frac{(\alpha+p\beta)\cdot\beta}{\beta^2}}\, E_{\alpha+p\beta}$$

$$\underline{-j}\, E_{\alpha-q\beta} = [\tfrac{1}{2}H_\beta, E_{\alpha-q\beta}] = \underline{\frac{(\alpha-q\beta)\cdot\beta}{\beta^2}}\, E_{\alpha-q\beta}$$

$$\Leftrightarrow \boxed{\begin{aligned} q+p &= 2j^0 \\ q-p &= \frac{2\alpha\cdot\beta}{\beta^2}\ (=n_2) \end{aligned}} \geq 0 \quad \Leftrightarrow \quad \begin{aligned} 2q &= 2j^0 + n_2 \\ p &= 2j^0 - n_2 \geq 0,\ q \geq p \end{aligned}$$

- in the special case $[E_\alpha, E_\beta] = 0$, i.e. $p = 0$

we have $q = 2j = n_2$, so the <u>$\beta$-string has length $n_2+1$</u>

(= multiplicity $2j+1$)

- a similar argument for the <u>$\alpha$-string</u> with $\beta + p'\alpha$, $\beta - q'\alpha$

gives for $p' = 0$ <u>a length of $n_1 + 1$</u>

example $r=2$:

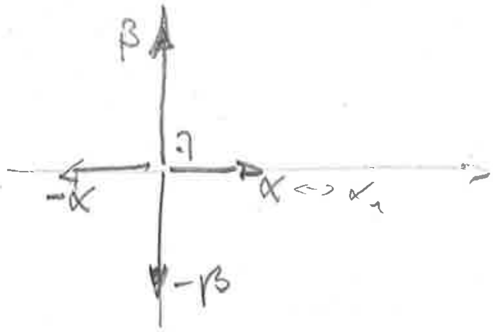
* in the case of <u>rank $r=2$</u> we can generate all possible roots from 2 roots $\alpha, \beta$, its $\alpha$- and $\beta$- strings as well as from root $\to$ $-$root and the exchange of $\alpha \leftrightarrow \beta$, <u>given the pair $n_1, n_2$</u> !

<u>let us discuss this in detail:</u>

a) <u>$n_1 = n_2 = 0$</u>

$\theta = \frac{\pi}{2}$

no length relation



this is $SO(4) \cong SU(2) \times SU(2)$ dim $\frac{4 \cdot 3}{2} = 6$, $r=2$

$\Rightarrow d - r = 4$ roots

we have to have

$[E_\alpha, E_\beta] = 0$

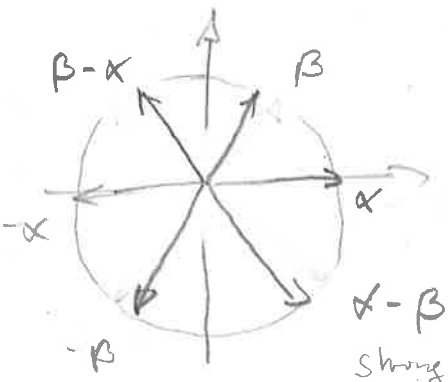
so $p = q = 0$

as the sum of 2 orthogonal roots would not have the right angle

b) <u>$n_1 = n_2 = 1$</u>

$\theta = \frac{\pi}{3}$

$|\alpha| = |\beta|$



$\alpha - \beta$ $(= -(\beta - \alpha))$

string of length $2 = n_1 + 1$

again the sum $\pm(\alpha + \beta)$ would not have the right length

$[E_\alpha, E_\beta] = 0$

this is $SU(3)$ dim $9 - 1 = 8$, $r = 2$

$\Rightarrow 8 - 2 = 6$ roots

c) $n_1 = 2,\ n_2 = 1$

$\theta = \frac{\pi}{4}$

$|\beta| = \sqrt{2}\,|\alpha|$

$\exists$ roots of 2 lengths

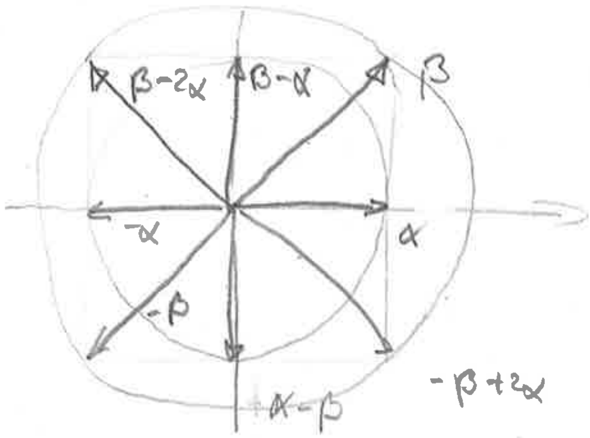
this is SO(5)

dim $\frac{5\cdot 4}{2} = 10$, $r=2$

$\Rightarrow 10-2 = 8$ roots

<u>$\alpha$-string</u> $\beta, \beta-\alpha, \beta-2\alpha$ has length $n_1+1$

<u>$\beta$-string</u> $\alpha, \alpha-\beta$ length $n_2+1$




$\alpha+\beta$ has wrong length & angle

$[E_\alpha, E_\beta] = 0$

d) $n_1 = 3,\ n_2 = 1$:

$\theta = \frac{\pi}{6}$

$|\beta| = \sqrt{3}\,|\alpha|$

$\exists$ roots of 2 length

this is $G_2$, an exceptional group

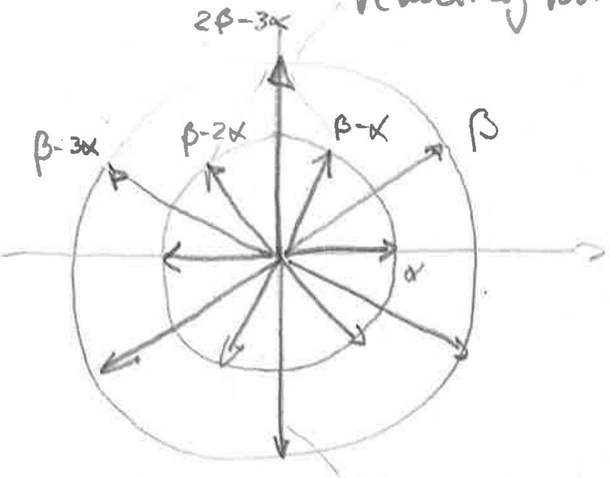
$r=2$, 12 roots

$\Rightarrow$ dim $d=14$

<u>$\alpha$-string:</u> $\beta, \beta-\alpha, \beta-2\alpha, \beta-3\alpha$ length $n_1+1$

<u>$\beta$-string:</u> $\alpha, \alpha-\beta$ length $n_2+1$

remaining root: has correct length & angle (from $\alpha \leftrightarrow \beta$ also)




* symmetries of these diagram: <u>Weyl-reflections</u> ($\to$ exercise) as an alternative way to construct roots

* complicated to draw for $r \geq 3$ $\rightsquigarrow$ better way: Dynkin diagram