Dynkin Diagrams

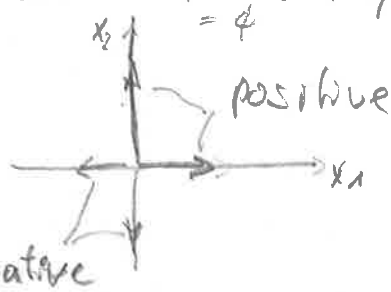
- problem: the roots are r-dim vectors, but we have $d-r$ of them for a rep. of dim d $\Rightarrow$ not all are indep!

  $\rightarrow$ how to find a good basis?

<u>Def</u>: A root is called <u>positive</u> (negative) if its first non-vanishing coefficient is positive (negative)

- obviously the set of positive roots P and the set of negative roots S don't intersect, $P \cap S = \phi$ (they depend on our choice of axis though) and $P \cup S$ gives all roots.

e.g. a) p. 97

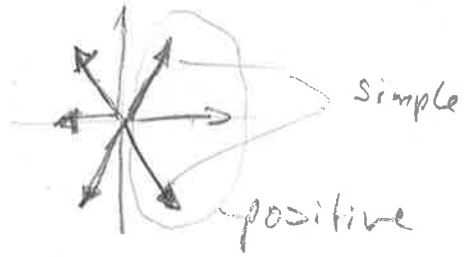


$\alpha = (\alpha_1, \alpha_2)$

- any negative root $= -$ positive root

<u>Def</u> A positive root is called <u>simple root</u>, if it cannot be expressed as a sum of positive roots.

e.g. b)



$\Rightarrow$ The simple roots form a <u>basis for the root diagram</u> in the following sense:

$\forall$ positive roots α $\underline{\exists\ n_i \geq 0 \text{ integers}}$ s.t. $\alpha = \sum_{\text{i all simple roots}} n_i \alpha_i$

where the α_i $(\neq (\alpha)_i)$ are the simple roots.

($\Rightarrow$ trivially the same holds for the negative roots with $n_i \leq 0$)

this is easy to see: * α simple ✓

* α positive, not simple $\Rightarrow \exists\, \beta, \gamma$ positive $\alpha = \beta + \gamma$
β, γ simple ✓, else continue. (Ends as we have a finite set)

$\Rightarrow$ the simple roots span all roots. To show that they form a basis we need to show: $\exists\ r$ of them and they are lin. indep.

- <u>for any two distinct simple roots</u> $\alpha_i, \alpha_j \Rightarrow \begin{cases} \alpha_i - \alpha_j \text{ is not a root} \\ \alpha_i \cdot \alpha_j \leq 0 \end{cases}$:

<u>Use</u> from Weyl-reflection (→ exercise)
$$\boxed{\begin{array}{l} \alpha \cdot \beta < 0 \Rightarrow \alpha + \beta \text{ is a root} \\ \alpha \cdot \beta > 0 \Rightarrow \alpha - \beta \text{ is a root} \end{array}}$$

Suppose $\alpha_i \cdot \alpha_j > 0 \Rightarrow \alpha_i - \alpha_j$ and $\alpha_j - \alpha_i$ are roots

$\alpha_i - \alpha_j$ positive $\Rightarrow \alpha_i = \alpha_j + (\alpha_i - \alpha_j)$ sum of pos. roots ↯

— " — negative $\Rightarrow \alpha_j = \alpha_i + (\alpha_j - \alpha_i)$ — " —

- <u>the simple roots are linearly indep</u>: show $\sum c_i \alpha_i = 0 \Rightarrow \forall_i\ c_i = 0$

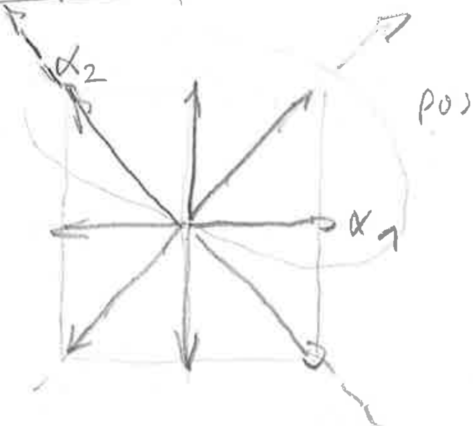
α_i positive roots $\Rightarrow$ some c_i have to be > 0, say p, some s $c_i = -b_i < 0$,

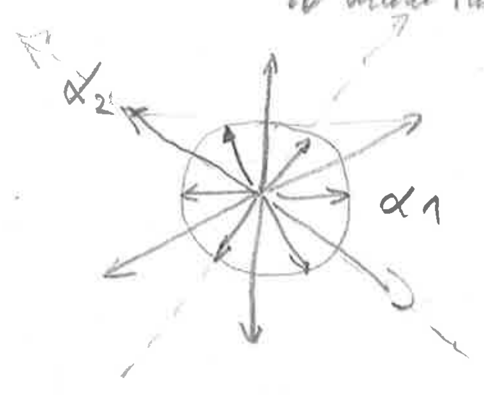
$$\Rightarrow \sum_i c_i \alpha_i = 0 \Leftrightarrow \sum_{i=1}^{s} c_i \alpha_i = \sum_{j=1}^{p} b_j \alpha_j \quad \text{, square}$$

$$\Rightarrow \underbrace{\left(\sum_i^s c_i \alpha_i\right)^2}_{>0} = \sum^s c_i \alpha_i \sum^p b_j \alpha_j = \sum_{i,j}^{s\,p} \underset{\geq 0}{c_i}\, \underset{\geq 0}{b_j}\, \underbrace{\alpha_i \cdot \alpha_j}_{\leq 0} \Rightarrow \text{all } c_i, b_j = 0$$

(we thus showed $\exists$ at max r simple roots, didn't show exactly r)

* <u>examples</u> for choice of basis = simple roots (choose axis accordingly to make them simple)

c) 

d) 

* the simple roots will be useful for a much simpler, pictorial classification of semi-simple Lie algebras.

Chevalley basis

If we only consider simple roots, we have the following subset of [,]s

p. 94 $[H_{\alpha_i}, E_{\alpha_j}] = \frac{2\alpha_i \cdot \alpha_j}{\alpha_i^2} E_{\alpha_j}$, $[E_{\alpha_i}, E_{-\alpha_j}] = \frac{1}{2}\alpha_i^2 \delta_{ij} H_{\alpha_i}$
as $N_{\alpha_i,-\alpha_j} = 0$ for $i \neq j$

The only disadvantage compared to the Cartan-Weyl basis is $(H_{\alpha_i}, H_{\alpha_j}) = \frac{4}{\alpha_i^2 \alpha_j^2} \alpha_i \cdot \alpha_j \neq \delta_{ij}$ analogous to p. 93

All essential info about the Lie-algebra is encoded in the

Def Cartan matrix $\boxed{K_{ij} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_j^2}}$, and in particular in its off-diagonal elements (since $\forall i$ $K_{ii} = 2$ (no sum)):

- because the simple roots form a basis $\det K \neq 0$ (neg signs because of angles of simple roots)

e.g. rank 2 : a) $SO(4)$: $K = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, c) $SO(5)$ $K = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$

b) $SU(3)$: $K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, d) G_2 $K = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

Def: Dynkin diagram : the info from K can be depicted as follows: for rank r we have r simple roots $\alpha_1 \ldots \alpha_r$, rep. by circles C_i. Two C_i, C_j are joined by $n_{ij} = K_{ij} K_{ji}$ (no sum) $= n_{ji}$ lines. When the length of 2 joined roots are different we put an arrow from the longer to the shorter root.

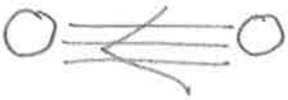
From our discussion on p. 94 we can only have $n_{ij} = 0, 1, 2, 3$.
with $n_1 = -K_{ij}$, $n_2 = -K_{ji}$

Example rank 2 : $n_{12} = n_1 n_2$

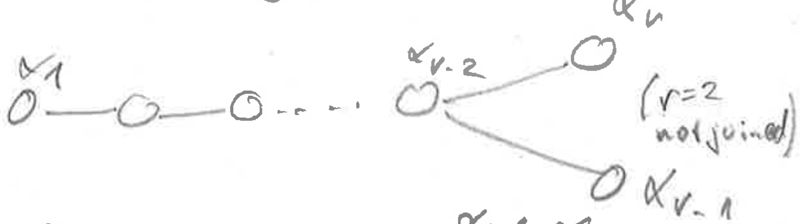
a) $n_{12}=0$  so(4) not joined (each O reps 1 SU(2))

b) $n_{12}=1$  su(3) $\alpha_{1,2}$ equal length

c) $n_{12}=2$  SO(5) 2 lengths for the roots

d) $n_{12}=3$  G_2 — " —

One can show that the following list is exhaustive, classifying all semi-simple Lie algebras (dig, H. Jones)

		alternative name
su(n), $r=n-1$		A_n
so(2n), $r=n$		D_n
so(2n+1) $r=n$		B_n
Sp(2n) $r=n$		C_n

* there are 5 so-called exceptional groups that cannot grow arbitrarily in length

G_2  , F_4 

E_6 

E_7 

E_8 

Addendum Dynkin Diagrams: Representations and weights

- the Cartan-Weyl basis and technique of Dynkin Diagrams was not only developed to classify compact semi-simple Lie-algebras but also to label representations

$\Rightarrow$ for that it will be useful that

- elements H_ℓ of the Cartan subalgebra act as analogues of J_3
- several E_α act as step operators $J_\pm$, see p. 93-94

*recall that a representation of SU(2) with spin j is given by

the set of eigenvectors to J_3: and

$$\boxed{\begin{aligned} J_3 |j,m\rangle &= m\,|j,m\rangle \quad , m=-j,\dots,+j \\ J_\pm |j,m\rangle &= \sqrt{(j\mp m)(j\pm m+1)}\,|j,m\pm 1\rangle \end{aligned}}$$

($\Leftrightarrow$ rep. is def. by action of algebra on a vector space)

in particular it holds $\underline{J_+|j,j\rangle = 0}$ (and $J_-|j,-j\rangle = 0$)

* given an algebra in Cartan-Weyl basis a representation can be labelled by the eigenvalues $\mu_{j=1,\dots,r}$, r rank of the algebra

$$\boxed{H_j|\mu\rangle = \mu_j|\mu\rangle}$$

where the r-component vector $\mu = (\mu_1,\dots,\mu_r)$ is called weight (vector).

It holds: $\underline{E_\alpha|\mu\rangle \sim |\mu+\alpha\rangle}$ for root $\alpha = (\alpha_1,\dots,\alpha_r)$ (of $[H_j, E_\alpha] = (\alpha)_j E_\alpha$)

as $H_j(E_\alpha|\mu\rangle) = (E_\alpha H_j + (\alpha)_j E_\alpha)|\mu\rangle = E_\alpha(\mu_j + (\alpha)_j)|\mu\rangle$

and hence $E_\alpha|\mu\rangle$ has eigenvalue $(\mu+\alpha)$

$\Rightarrow$ acting by a different $E_{\beta\neq\alpha}$ we obtain $E_\beta E_\alpha|\mu\rangle \sim |\mu+\alpha+\beta\rangle$ etc.