Addendum Dynkin Diagrams: Representations and weights

- the Cartan-Weyl basis and technique of Dynkin Diagrams was not only developped to classify compact semi-simple Lie-algebras, but also to label representations

→ for that it will be useful that
- elements H_ℓ of the Cartan subalgebra act as analogues of J_3
- several E_α act as stepoperators $J_\pm$, see p. 83-84

*recall that a representation of $SU(2)$ with spin j is given by

the set of eigenvectors to J_3:
$$\boxed{\begin{aligned} &J_3 |j,m\rangle = m|j,m\rangle, \quad m=-j,\dots,+j \\ \text{and} \quad &J_\pm |j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\,|j,m\pm 1\rangle \end{aligned}}$$

($\Leftrightarrow$ rep. is def. by action of algebra on a vector space)

in particular it holds $\quad \underline{J_+|j,j\rangle = 0}$ (and $J_-|j,-j\rangle = 0$)

* given an algebra in Cartan-Weyl basis a representation can be labelled by the eigenvalues $\mu_{j=1,\dots,r}$, r rank of the algebra

$$\boxed{H_j|\mu\rangle = \mu_j|\mu\rangle}$$

where the r-component vector $\mu = (\mu_1,\dots,\mu_r)$ is called weight (vector).

It holds: $\quad \underline{E_\alpha|\mu\rangle \sim |\mu+\alpha\rangle}$ for root $\alpha = (\alpha_1,\dots,\alpha_r)$ (of $[H_j, E_\alpha] = (\alpha)_j E_\alpha$)

as $H_j(E_\alpha|\mu\rangle) = (E_\alpha H_j + (\alpha)_j E_\alpha)|\mu\rangle = E_\alpha(\mu_j + (\alpha)_j)|\mu\rangle$

and hence $E_\alpha|\mu\rangle$ has eigenvalue $(\mu+\alpha)$

⇒ acting by a different $E_{\beta\neq\alpha}$ we obtain $E_\beta E_\alpha|\mu\rangle \sim |\mu+\alpha+\beta\rangle$ etc.

$\Rightarrow$ the weights μ of this rep. lie on a discrete lattice spanned by the different roots, the so-called weight lattice.

$\Rightarrow$ The root lattice (see e.g. p. 97–98 for rank 2) is a subset of the weight lattice, obtained by adding roots to the origin.

Q: where does a particular rep lie on the weight lattice, i.e. by adding roots to which starting point?

* recall that $H_\alpha = \frac{2\alpha \cdot H}{\alpha^2}$ acts as $J_3 \sim \frac{1}{2}H_\alpha$ of an $su(2)$ subalgebra

so H_α has integer eigenvalues (see p. 94)

$\rightarrow$ applied to the rep $|\mu\rangle$ we have $H_\alpha |\mu\rangle = 2\frac{\alpha \cdot H}{\alpha^2}|\mu\rangle = \frac{2\alpha\cdot\mu}{\alpha^2}|\mu\rangle$

$$\Rightarrow \quad \frac{2\alpha\cdot\mu}{\alpha^2} = n \in \mathbb{Z} \quad \text{is an integer}$$

• if we shift μ by β using a different step operator E_β we get that $|\mu+\beta\rangle \sim E_\beta|\mu\rangle$ has eigenvalue

$$\frac{2\alpha\cdot(\mu+\beta)}{\alpha^2} = n' \in \mathbb{Z} \quad \text{integer} \qquad \left(\text{due to } [H_\alpha, E_\beta] = \frac{2\alpha\cdot\beta}{\alpha^2}E_\beta \text{ p. 94}\right)$$

this is indeed satisfied as also $\frac{2\alpha\cdot\beta}{\alpha^2} \in \mathbb{Z}$

* recall: for the roots of an alg. we introduced a basis, the simple roots α_i (among all positive roots, see p. 95)

Def: the basis for the weights is given by the fundamental weights λ_i:

$$\text{defined by} \quad \boxed{\frac{2\lambda_i\cdot\alpha_j}{\alpha_j^2} = \delta_{ij}} \qquad i,j = 1,\dots,r \text{ (we had } r \text{ simple roots)}$$

$\Rightarrow$ all weights μ can be expanded in terms of the λ_i:

$$\boxed{\mu = \sum_{i=1}^{r} m_i \lambda_i}$$ with $m_i \in \mathbb{Z}$, or by projection

$$\frac{2\alpha_j \cdot \mu}{\alpha_j^2} = m_j$$

* from now on consider only step operators E_{α_i} for simple roots α_i:

For a finite-dim rep. the repeated action of a step op. on the states of the rep must terminate for some $|\mu\rangle$:

$$E_{\alpha_i}|\mu\rangle = 0$$ (just as $J_+|j,j\rangle = 0$ for a single $su(2)$)

Def: The highest weight state $|\mu\rangle$ is the state annihilated by all step ops of simple roots E_{α_i}:

$$\boxed{E_{\alpha_i}|\mu\rangle = 0 \quad \text{for } i = 1, 2, \ldots, r}$$

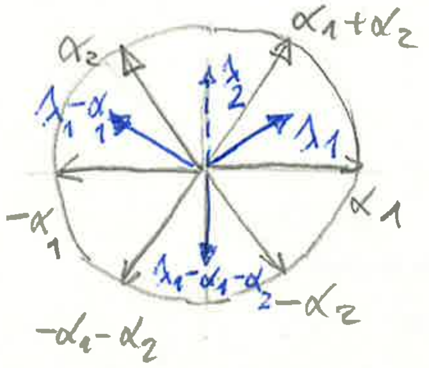
* It is the equivalent to $|j,j\rangle$ in $su(2)$

$\rightarrow$ the labels of the highest weight $|\mu\rangle$ label the entire rep, and the remaining states of the rep are obtained by the action (repeated) of lowering ops. $E_{-\alpha_i}$

* Weights can be ordered according to their expansion coeffs m_i, by choosing the first $m_i \neq 0$ to do the ordering.

Thm Dynkin: showed that $|\mu\rangle$ is highest weight iff $\forall_{j=1,\ldots,r}\ m_j \geq 0$.

Example $su(3)$



Simple roots: α_1, α_2 rank $r=2$

* note that $\alpha_1-\alpha_2$ is not a root! $\Rightarrow [E_{\alpha_1}, E_{-\alpha_2}]=0$

Construction of fundamental weights λ_1, λ_2

satisfy $\boxed{\dfrac{2\lambda_i \cdot \alpha_j}{\alpha_j^2} = \delta_{ij}}$

- in Cartesian coordinates

we have $\alpha_1 = (1,0) \Rightarrow \alpha_1^2 = 1$

$\alpha_2 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \Rightarrow \alpha_2^2 = 1$

Solution: $\boxed{\begin{aligned}\lambda_1 &= \left(\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}\right) \\ \lambda_2 &= \left(0, \tfrac{1}{\sqrt{3}}\right)\end{aligned}}$ $\perp \alpha_2$ ✓ $\perp \alpha_1$ ✓

$\Rightarrow$ general weight $\boxed{\mu = m_1\lambda_1 + m_2\lambda_2}$: $(m_1, m_2)_w$

- Simplest example for the highest weight of an irrep:

$m_1 = 1, m_2 = 0$ (which turns out to be the fundamental irrep 3)

$\to$ generate all other states of this irrep by acting with $E_{-\alpha_1}, E_{-\alpha_2}$

* for a given, general weight $|\mu\rangle$ the step op $E_{\pm\beta}$ will produce a series of other weights up to $|\mu+p\beta\rangle$, $|\mu-q\beta\rangle$, $p, q \geq 0$.

$\to$ the same argument as for the roots on page 96 lead to

$\boxed{\begin{aligned} q+p &= 2j \\ q-p &= \frac{2\mu\cdot\beta}{\beta^2}\end{aligned}}$ where $j \in \frac{\mathbb{N}}{2}$ labels the length of the series of weights $2j+1$

Here: $|\lambda_1\rangle$ is highest weight, so $E_{+\alpha_1}|\lambda_1\rangle = 0$ and $E_{+\alpha_2}|\lambda_1\rangle = 0$

so $p=0$ for both α_1 and α_2.

$\Rightarrow$ • for α_1 and the corresp. $SU(2)_{\alpha_1}\{H_{\alpha_1}, E_{+\alpha_1}, E_{-\alpha_1}\}$ the length of the series generated by $E_{-\alpha_1}$ is : $q = 2j \Rightarrow$

$$2j + 1 = q + 1 = \frac{2\Lambda_1\alpha_1}{\alpha_1^2} + 1 = m_1 + 1 = 2$$

$\Rightarrow$ we have states $|\Lambda_1\rangle$ and $E_{-\alpha_1}|\Lambda_1\rangle \sim |\Lambda_1 - \alpha_1\rangle$ in our irrep (which forms an $SU(2)_{\alpha_1}$ doublet with spin $\frac{1}{2}$)

• for α_2 the length of the series is $q = 2j \Rightarrow$

$$2j + 1 = q + 1 = \frac{2\Lambda_1\alpha_2}{\alpha_2^2} + 1 = m_2 + 1 = 1$$

$\Rightarrow$ we only have $|\Lambda_1\rangle$ and $E_{-\alpha_2}|\Lambda_1\rangle = 0$,

(this is a singlet $j = 0$ of $SU(2)_{\alpha_2}$)

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* we have exhausted the application of step op. $E_{\pm\alpha_1}, E_{\pm\alpha_2}$ on $|\Lambda_1\rangle$, let's continue to operate on $|\Lambda_1 - \alpha_1\rangle$

• we already saw $E_{-\alpha_1}|\Lambda_1 - \alpha_1\rangle = 0$, and $E_{+\alpha_1}$ brings back $|\Lambda_1\rangle$

• $E_{+\alpha_2}|\Lambda_1 - \alpha_1\rangle \sim E_{+\alpha_2}E_{-\alpha_1}|\Lambda_1\rangle = E_{-\alpha_1}E_{+\alpha_2}|\Lambda_1\rangle = 0$ so $p = 0$

as $[E_{\alpha_2}, E_{\alpha_1}] = 0$, $-\alpha_1 + \alpha_2$ is not a root

$|\Lambda_1 - \alpha_1\rangle$ has $(m_1, m_2)_w = (-1, 1) \Rightarrow 2j + 1 = m_2 + 1 = 2$
and $q = 1$

$\Rightarrow E_{-\alpha_2}|\Lambda_1 - \alpha_1\rangle \sim |\Lambda_1 - \alpha_1 - \alpha_2\rangle \neq 0$

and $|\Lambda_1 - \alpha_1\rangle, |\Lambda_1 - \alpha_1 - \alpha_2\rangle$ form doublet under $SU(2)_{\alpha_2}$

• $|\Lambda_1 - \alpha_1 - \alpha_2\rangle$ is a singlet under $SU(2)_{\alpha_1}$ as $E_{+\alpha_1}|\Lambda_1 - \alpha_1 - \alpha_2\rangle = 0$
(as $|\Lambda_1 - \alpha_2\rangle$ is not a state in this rep)

• all further action of step op on $|\Lambda_1 - \alpha_1 - \alpha_2\rangle$ gives zero

((hypercharge $Y = (m_1 + 2m_2)/3$

$\Rightarrow$ this irrep consists of 3 states : $\left\{ \begin{array}{l} |\Lambda_1\rangle \ (\hat{=}\, u),\ |\Lambda_1 - \alpha_1\rangle \ (\hat{=}\, d) \\ |\Lambda_1 - \alpha_1 - \alpha_2\rangle \ (\hat{=}\, s) \end{array} \right\} \Leftarrow$ dim 3

called the quark triplet rep: $\overset{m_1=1}{\underset{\Lambda_1}{\circ}} \!-\!\!-\!\!-\! \overset{m_2=0}{\underset{\Lambda_2}{\circ}}$

- the same procedure starting from Λ_2 as highest weight, or in other words $(m_1, m_2)_w = (0,1)_w$ leads to the antiquark triplet irrep $\bar{3}$, with $\overset{0}{\circ}\!-\!\overset{1}{\circ}$

- the irrep with highest weight $\Lambda_1 + \Lambda_2$ $= \alpha_1 + \alpha_2$ or $(m_1, m_2)_w = (1,1)$ is the adjoint rep: $\overset{1}{\circ}\!-\!\overset{1}{\circ}$
  it has all roots as weights (states) plus two degenerate weights $\mu = 0$ ($\Rightarrow$ the dim is 8 of this irrep)

- other highest weight (tensor) irreps of $SU(3)$ $(m_1, m_2)_w$ $\overset{m_1}{\circ}\!-\!\overset{m_2}{\circ}$
  [see H. Georgi, Lie Algebras in Particle Physics, 1982]
  have dimension $\frac{1}{2}(m_1+1)(m_2+1)(m_1+m_2+2)$

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* so far we have found all weights (or states) in an irrep starting from the highest weight $|\mu\rangle$.

$\rightarrow$ In order to determine the actual rep. matrices we need to determine the coefficients in $E_{-\alpha}|\mu\rangle \sim_{\uparrow} |\mu-\alpha\rangle$, the analogues to $\sqrt{(j\mp m)(j\pm m+1)}$ in $SU(2)$. Sandwiches of all states will then give the matrix elements of the rep. matrices.

Define the (complex) coefficients $N_{-\alpha,\mu}$ for weight μ and simple root α

by $$E_{-\alpha}|\mu\rangle \equiv N_{-\alpha,\mu}|\mu-\alpha\rangle \quad \overset{\langle\mu-\alpha|\cdot}{\Longleftrightarrow} \quad \boxed{N_{-\alpha,\mu} \equiv \langle\mu-\alpha|\big(E_{-\alpha}|\mu\rangle\big)}$$

where we assumed that $\langle\mu-\alpha|\mu-\alpha\rangle = 1$ is normalized.

- consider $$\langle\mu|[E_\alpha, E_{-\alpha}]|\mu\rangle = \langle\mu|(E_\alpha E_{-\alpha} - E_{-\alpha}E_\alpha)|\mu\rangle$$
$$\overset{A}{=} \langle\mu|E_{-\alpha}^\dagger E_{-\alpha}|\mu\rangle - \langle\mu|E_\alpha^\dagger E_\alpha|\mu\rangle = |N_{-\alpha,\mu}|^2 - |N_{\alpha,\mu}|^2$$

$SU(n)$ $\quad E_{-\alpha} = E_\alpha^\dagger$ p. 91