- the same procedure starting from Λ_2 as highest weight, or in other words $(m_1, m_2)_w = (0,1)_w$ leads to the antiquark triplet irrep $\bar{3}$, with $\overset{0}{\circ}\!-\!\overset{1}{\circ}$

- the irrep with highest weight $\Lambda_1 + \Lambda_2$ or $(m_1, m_2)_w = (1,1)$ is the adjoint rep: $= \alpha_1 + \alpha_2$ $\quad \overset{1}{\circ}\!-\!\overset{1}{\circ}$
it has all roots as weights (states) plus two degenerate weights $\mu = 0$ ($\Rightarrow$ the dim is 8 of this irrep)

- other highest weight (tensor) irreps of $SU(3)$ $(m_1, m_2)_w$ $\overset{m_1}{\circ}\!-\!\overset{m_2}{\circ}$
[see H. Georgi, Lie Algebras in Particle Physics, 1982]
have dimension $\frac{1}{2}(m_1+1)(m_2+1)(m_1+m_2+2)$

Representation matrices:

$\star$ so far we have found all weights (or states) in an irrep starting from the highest weight $|\mu\rangle$.

$\rightarrow$ In order to determine the actual rep. matrices we need to determine the coefficients in $E_{-\alpha}|\mu\rangle \underset{A}{\sim} |\mu-\alpha\rangle$, the analogues to $\sqrt{(j\mp m)(j\pm m+1)}$ in $SU(2)$. Sandwiches of all states will then give the matrix elements of the rep. matrices.

Define the (complex) coefficients $N_{-\alpha,\mu}$ for a weight μ and $\overset{\text{simple}}{\text{root}}$ α

by $E_{-\alpha}|\mu\rangle \equiv N_{-\alpha,\mu}|\mu-\alpha\rangle \overset{\langle\mu-\alpha|\cdot}{\Longleftrightarrow} \boxed{N_{-\alpha,\mu} \equiv \langle\mu-\alpha|\left(E_{-\alpha}|\mu\rangle\right)}$

where we assumed that $\langle\mu-\alpha|\mu-\alpha\rangle = 1$ is normalized.

- consider $\langle\mu|[E_\alpha, E_{-\alpha}]|\mu\rangle = \langle\mu|(E_\alpha E_{-\alpha} - E_{-\alpha}E_\alpha)|\mu\rangle$

$$\underset{A}{=} \langle\mu|E_{-\alpha}^\dagger E_{-\alpha}|\mu\rangle - \langle\mu|E_\alpha^\dagger E_\alpha|\mu\rangle = |N_{-\alpha,\mu}|^2 - |N_{\alpha,\mu}|^2$$

$E_\alpha = E_{-\alpha}^\dagger$ p. 91

on the other hand $[E_\alpha, E_{-\alpha}] = \alpha\cdot H$ and $H_i|\mu\rangle = \mu_i|\mu\rangle$

$\Rightarrow$ $\langle\mu|[E_\alpha, E_{-\alpha}]|\mu\rangle = \langle\mu|\alpha\cdot H|\mu\rangle = \alpha\cdot\mu = |N_{-\alpha,\mu}|^2 - |N_{\alpha,\mu}|^2$

• we also have $N^*_{-\alpha,\mu} = \langle\mu|E^+_{-\alpha}|\mu-\alpha\rangle = \langle\mu|E_\alpha|\mu-\alpha\rangle = N_{\alpha,\mu-\alpha}$

$$\Rightarrow \boxed{|N_{\alpha,\mu-\alpha}|^2 - |N_{\alpha,\mu}|^2 = \alpha\cdot\mu}$$

(e.g. the highest weight)

• if we start with a weight (state) $|\mu\rangle$ annihilated by E_α: $N_{\alpha,\mu}=0$

$\Rightarrow E_\alpha|\mu\rangle = 0$, we generate the following ~~string~~ with $E_{-\alpha}$'s:

$|\mu\rangle, \dots, |\mu - q\alpha\rangle$ with $q = \frac{2\mu\cdot\alpha}{\alpha^2}$. This leads to

the set of difference eqs.:

$$\left.\begin{aligned}
|N_{\alpha,\mu-\alpha}|^2 - 0 &= \alpha\cdot\mu\\
|N_{\alpha,\mu-2\alpha}|^2 - |N_{\alpha,\mu-\alpha}|^2 &= \alpha\cdot(\mu-\alpha)\\
&\vdots\\
|N_{\alpha,\mu-q\alpha}|^2 - |N_{\alpha,\mu-(q-1)\alpha}|^2 &= \alpha\cdot(\mu-(q-1)\alpha)\\
0 - |N_{\alpha,\mu-q\alpha}|^2 &= \alpha\cdot(\mu-q\alpha)
\end{aligned}\right\}\ \text{sum}$$

with solution

$$|N_{\alpha,\mu-t\alpha}|^2 = -\sum_{s=t}^{q} \alpha\cdot(\mu - s\alpha) = -(q-t+1)\alpha\cdot\mu + \alpha\cdot\alpha\sum_{s=t}^{q} s$$

$$= -\tfrac{1}{2}(q-t+1)\left(2\alpha\cdot\mu - (q+t)\alpha^2\right)$$

$$= \tfrac{1}{2}\alpha^2 t(q-t+1) \quad \text{for } t = 1, 2, \dots, q$$

• $q = 2j$, $t = j-m$ $\Rightarrow$ $\sim (j-m)(j+m+1)$ ok for $SU(2)$

• phases of system of eqs subtle to fix. • $N_{\alpha\beta}$ in Cartan-Weyl basis $\Leftrightarrow$ $N_{\alpha,\mu}$ in adj. rep.!

The Group SU(N) : Irreducible Representations and Young Tableaux

- $\underline{SU(2)}$:

the fundamental, irreducible rep of $SU(2)$ (2x2 matrix rep)
acts on 2-component spinors ψ_a, with $a=1,2$ corresponding
to $m = \pm\frac{1}{2}$. The transformation under the action of an $SU(2)$ matrix
is then given by
$$\underline{\psi'_a = U_a{}^b \psi_b}, \quad U \in SU(2)$$

* we have already seen (for $SU(N)$ in general) that with U also
its complex conjugate is a representation, $D(U) = U^*$.
We will denote (2-component) spinors transforming under U^* by upper indices:
$$\underline{\psi'^a = U^{*a}{}_b \psi^b}$$

* while in general $SU(N)$ the reps U and U^* are different,
we will show now that in the special case of $SU(2)$ they
are equivalent. The reason is that for $SU(2)$ complex
conjugation can be represented as a group action:

recall that the Pauli-matrices are the generators of $SU(2)$ and
$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
(-> p. 67)
$$\underline{U = \exp[-\tfrac{1}{2} i \vec{\sigma}\cdot\vec{\alpha}]}, \quad \vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \quad \text{for } U \in SU(2)$$

with $[\sigma_j, \sigma_k] = 2i\,\varepsilon_{jkl}\,\sigma_l$, and in particular
$$\sigma_j \sigma_k = i\,\varepsilon_{jkl}\,\sigma_l \quad (\text{i.e. } \sigma_1\sigma_2 = i\sigma_3 \text{ etc})$$
$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \mathbb{1}$$

consider $\quad \underline{C \equiv i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} \Leftrightarrow (C)^{ab} = \varepsilon^{ab}$,

it holds $C^\dagger = C^T = -C = C^{-1}$, $C^2 = -\mathbb{1}$, $\quad C^{-1} = -i\sigma_2$ as $\sigma_2^2 = \mathbb{1}$
so C unitary

- $\underline{\text{conjugation of}}$ the 3 Pauli matrices with C yields

$$C\sigma_1 C^{-1} = i\sigma_2 \sigma_1 (-i\sigma_2) = i\sigma_2\, i(-i)\sigma_3 = -\sigma_1$$

$$C\sigma_2 C^{-1} = i\sigma_2 \sigma_2 (-i\sigma_2) = +\sigma_2$$

$$C\sigma_3 C^{-1} = i\sigma_2 \sigma_3 (-i\sigma_2) = i^2 \sigma_1 (-i\sigma_2) = -\sigma_3$$

- equivalently $\quad \boxed{C\vec{\sigma}\, C^{-1} = -\vec{\sigma}^*}$

applied to the above rep of an $SU(2)$ matrix (with $\vec{\alpha} \equiv \vec{n}\,\theta$)

we have
$$C U C^{-1} = C \exp[-\tfrac{i}{2}\vec{\sigma}\cdot\vec{n}\,\theta]\, C^{-1}$$
$$= C\left(\mathbb{1}\cos\tfrac{1}{2}\theta - i\vec{\sigma}\cdot\vec{n}\sin\tfrac{\theta}{2}\right) C^{-1}$$
$$= \mathbb{1}\cos\tfrac{1}{2}\theta + i\vec{\sigma}^*\cdot\vec{n}\sin\tfrac{\theta}{2} \quad = U^*$$

$$\Leftrightarrow \quad \boxed{CUC^{-1} = U^*}$$

in other words the map $U \to U^*$ (complex conjugation) is a $\underline{\text{group automorphism}}$. Below we will also show that C maps ψ_a to the complex conj rep: $\psi^a = C^{ab}\psi_b$.

---

* $\quad \varepsilon$ is an invariant $\underline{\text{tensor}}$ under $SU(2)$:

$$\varepsilon'_{cd} = U_c{}^a U_d{}^b \varepsilon_{ab} \stackrel{!}{=} \varepsilon_{cd} \qquad \text{tensor trafo}$$

this follows from the above using $U^{-1} = U^\dagger = U^{*T}$ and $C^2 = -\mathbb{1}$:

$C^{-1} = -C \Rightarrow \quad C U C^{-1} = U^* \Leftrightarrow C^{-1} U C = (U^{-1})^T$

$\cdot U^T | \qquad \Rightarrow C^{-1} U C U^T = \mathbb{1}$

$C \cdot | \qquad \Rightarrow \boxed{U C U^T = C}$

* we also have from $C^* = C$: $\Leftrightarrow U^* C U^\dagger = C \quad | \cdot U$

$\Rightarrow \underbrace{U^* C = C U}_{(*)} \; | U^T \cdot \quad \Rightarrow U^T U^* C = \boxed{C = U^T C U}$

- back to the invariance: we will rather show that

$$\varepsilon'_{cd} = \det U \; \varepsilon_{cd}$$

, and thus due to $U \in SU(2)$ $\Rightarrow \det U = 1$ invariance

check:

$\varepsilon'_{11} = U_1{}^1 U_1{}^2 - U_1{}^2 U_1{}^1 = 0 = \det U \cdot \varepsilon_{11}^{=0}$

$\varepsilon'_{12} = U_1{}^1 U_2{}^2 - U_1{}^2 U_2{}^1 = \det U \cdot \varepsilon_{12}^{=+1}$

$\varepsilon'_{21} = U_2{}^1 U_1{}^2 - U_2{}^2 U_1{}^1 = -\det U = \det U \cdot \varepsilon_{21}^{=-1}$

$\varepsilon'_{22} = U_2{}^1 U_2{}^2 - U_2{}^2 U_2{}^1 = 0 = \det U \cdot \varepsilon_{22}^{=0}$ ☑

- we will now show that $\underline{\psi^a = C^{ab} \psi_b \text{ transforms}}$ wrt U^* and is thus a spinor with upper index:

$$\underline{\psi^{a\prime}} = C^{ab} \psi'_b = C^{ab} U_b{}^c \psi_c = (CU)^{ac} \psi_c \overset{(*)}{=} (U^* C)^{ac} \psi_c$$

$$= \underline{U^{*a}{}_d \, C^{db} \psi_b}$$

* the ε-tensor (or matrix C) thus acts like a (antisym) metric, by lowering and raising indices (as $g_{\mu\nu}$ or $\eta_{\mu\nu}$ in relativity).

$$\psi_b = (C^{-1})_{ba} \psi^a$$

→ we can use the epsilon tensor to generate invariants:

Invariants:

- $\psi^a\varphi_a = \varepsilon^{ab}\psi_b\varphi_a$ is invariant under SU(2) trafos for general spinors ψ_a, φ_a and thus is a scalar quantity:

$$
\begin{aligned}
\psi'^a\varphi'_a = \varepsilon^{ab}\psi'_b\varphi'_a
&= \varepsilon^{ab}U_b{}^d\psi_d\,U_a{}^c\varphi_c = (U^TCU)^{cd}\psi_d\varphi_c \\
&= C^{cd}\psi_d\varphi_c = \varepsilon^{cd}\psi_d\varphi_c = \psi^c\varphi_c
\end{aligned}
$$

- a generic tensor $T^{a_1a_2\ldots}{}_{b_1b_2\ldots}$ transforms w.r.t. U^* for the upper and wrt U for the lower indices. Any index pair contracted with ε^{ab} does not transform:

$$T'^{a_1a_2\ldots a_k}{}_{b_1\ldots b_l} = U^{*a_1}{}_{c_1}\ldots U^{*a_k}{}_{c_k}\,U_{b_1}{}^{d_1}\ldots U_{b_l}{}^{d_l}\,T^{c_1\ldots c_k}{}_{d_1\ldots d_l}$$

→ by contracting spinors and tensors appropriately we can build other invariants

Higher-dimensional tensor irreps & Clebsch-Gordon coefficients

- the product of the two spinors ψ_a and φ_b (each forming an SU(2) irrep) can be combined into 4 different ways:

$\psi_a\varphi_b$, $a,b=1,2$:

$$\left.\begin{array}{l}\psi_1\varphi_1\\ \psi_2\varphi_2\\ \psi_1\varphi_2+\psi_2\varphi_1\end{array}\right\}\ \text{symmetric in indices} \quad \text{spin 1 multiplett}$$

$$\left.\psi_1\varphi_2-\psi_2\varphi_1 = \varepsilon^{cd}\psi_c\varphi_d\ \right\}\ \text{antisym. in indices, is invariant under SU(2)} \Leftrightarrow \text{has spin 0}$$

as a Clebsch-Gordon series (see p. 79)

this is $\underset{\text{SU(2) rep with } \ell=\frac12}{\frac12\otimes\frac12} = 1\oplus 0 \left(=\sum_{\ell=|l_1-l_2|}^{l_1+l_2}\oplus\, a_\ell D^{(\ell)}\right)$ ((sometimes the dim $2j+1$ of the rep is used instead: $2\otimes 2 = 3\oplus 1$))

defining $\psi_{(a}\varphi_{b)} \equiv \psi_a\varphi_b + \psi_b\varphi_a$ symmetrisation (also as $\psi_a\varphi_b$)

$\psi_{[a}\varphi_{b]} \equiv \psi_a\varphi_b - \psi_b\varphi_a$ anti-symmetrisation (also as $\psi_a\varphi_b$)

we have $\psi_a\varphi_b = \frac{1}{2}\psi_{(a}\varphi_{b)} + \frac{1}{2}\psi_{[a}\varphi_{b]}$

which can be represented pictorially as the following Young-tableau

$$\boxed{a} \times \boxed{b} = \boxed{a\,|\,b} + \boxed{\begin{array}{c} a \\ b \end{array}}$$

Example from particle physics — mesons:

- the two lightest quarks u and d (often taken to be massless as an approximation) form a so-called isospin doublet

$$q = \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \text{ with } \begin{array}{lll} u\ (\equiv \psi_1) & \text{having} & I_3 = +\frac{1}{2} \text{ under } SU(2)_I \\ d\ (\equiv \psi_2) & \text{"} & I_3 = -\frac{1}{2} \end{array}$$

- in order to form colour neutral objects we may combine quark - antiquark pairs in the following way:

anti particles are rep. by the complex conjugate wave function (denoted by $^{-}$)

i.e. $\bar{q}^1 = \bar{u}$, $\bar{q}^2 = \bar{d}$ $(\hat{=}\ \varphi\text{'s})$

$$\Rightarrow \bar{q}_a = (C^{-1})_{ab}\,\bar{q}^b = -\varepsilon_{ab}\,\bar{q}^b, \text{ so } \bar{q} = \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix} = \begin{pmatrix} \bar{q}_1 \\ \bar{q}_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

- the spinless, invariant combination is

$$\text{"}\eta\text{"} = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) \quad \text{(normalised to unity)}$$

particle physics: the observed η and η' also contain $\pm s\bar{s}$

whereas the spin 1 multiplett is

$$\pi^+ = -u\bar{d}$$
$$\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$
$$\pi^- = d\bar{u}$$