Invariants:

- $\underline{\psi^a \varphi_a = \varepsilon^{ab} \psi_b \varphi_a}$ is invariant under SU(2) trafos for general spinors ψ_a, φ_a and thus is a scalar quantity:

$$\underline{\psi'^a \varphi'_a} = \varepsilon^{ab} \psi'_b \varphi'_a$$
$$= \varepsilon^{ab} U_b{}^d \psi_d U_a{}^c \varphi_c = (U^T C U)^{cd} \psi_d \varphi_c$$
$$= C^{cd} \psi_d \varphi_c = \varepsilon^{cd} \psi_d \varphi_c = \underline{\psi^c \varphi_c}$$

- a generic tensor $T^{a_1 a_2 ..}{}_{b_1 b_2 ..}$ transforms w.r.t. U^* for the upper and w.r.t. U for the lower indices. Any index pair contracted with ε^{ab} does not transform:

$$T'^{a_1 a_2 .. a_k}{}_{b_1 .. b_\ell} = U^{*a_1}{}_{c_1} .. U^{*a_k}{}_{c_k} U_{b_1}{}^{d_1} .. U_{b_\ell}{}^{d_\ell} T^{c_1 .. c_k}{}_{d_1 .. d_\ell}$$

$\to$ by contracting spinors and tensors appropriately we can build other invariants

Higher-dimensional tensor irreps & Clebsch-Gordon coefficients

- the product of the two spinors ψ_a and φ_b (each forming an SU(2) irrep) can be combined into 4 different ways:

$\psi_a \varphi_b$, $a, b = 1, 2$:

$$\left.\begin{array}{l} \psi_1 \varphi_1 \\ \psi_2 \varphi_2 \\ \psi_1 \varphi_2 + \psi_2 \varphi_1 \end{array}\right\} \text{symmetric in indices} \quad \text{spin 1 multiplett}$$

$$\psi_1 \varphi_2 - \psi_2 \varphi_1 = \varepsilon^{cd} \psi_d \varphi_c \Big\} \text{ antisym. in indices, is invariant under SU(2)} \Leftrightarrow \text{has spin 0}$$

as a Clebsch-Gordon series (see p. 79)

this is $\underset{\text{SU(2) rep with } \ell = 1/2}{\frac{1}{2} \otimes \frac{1}{2}} = 1 \oplus 0 \left(= \sum_{\ell = |\ell_1 - \ell_2|}^{\ell_1 + \ell_2} \oplus a_\ell D^{(\ell)} \right)$ ((sometimes the dim $2j+1$ of the rep is used instead: $2 \otimes 2 = 3 \oplus 1$

defining $\psi_{(a}\varphi_{b)} \equiv \psi_a\varphi_b + \psi_b\varphi_a$ symmetrisation (also as $\psi_a\varphi_b$)

$\psi_{[a}\varphi_{b]} \equiv \psi_a\varphi_b - \psi_b\varphi_a$ anti-symmetrisation (also as $\psi_a\varphi_b$)

we have $\psi_a\varphi_b = \frac{1}{2}\psi_{(a}\varphi_{b)} + \frac{1}{2}\psi_{[a}\varphi_{b]}$

which can be represented pictorially as the following Young-tableau

$$\boxed{a}\times\boxed{b} = \boxed{a\,|\,b} + \boxed{\begin{smallmatrix}a\\ b\end{smallmatrix}}$$

sym index pair   antisym index pair

Example from particle physics — mesons:

- the two lightest quarks u and d (often taken to be massless as an approximation) form a so-called Isospin doublet

$$q = \begin{pmatrix}u\\ d\end{pmatrix} = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}a_1\\ a_2\end{pmatrix} \text{ with } \begin{array}{lll} u\,(\equiv\psi_1) & \text{having} & I_3 = +\frac{1}{2} \text{ under } SU(2)_I \\ d\,(\equiv\psi_2) & \text{"} & I_3 = -\frac{1}{2} \end{array}$$

- in order to form colour neutral objects we may combine quark-antiquark pairs in the following way:

anti particles are rep. by the complex conjugate wave function (denoted by $\bar{\ }$)

i.e. $\bar q^1 = \bar u$, $\bar q^2 = \bar d$ ($\hat{=}\ \varphi$'s)

$$\Rightarrow \bar q_a = (\epsilon^{-1})_{ab}\,\bar q^b = -\epsilon_{ab}\,\bar q^b, \text{ so } \bar q = \begin{pmatrix}-\bar d\\ \bar u\end{pmatrix} = \begin{pmatrix}\bar q_1\\ \bar q_2\end{pmatrix} = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$

- the spinless, invariant combination is

$$\text{"}\eta\text{"} = \frac{1}{\sqrt{2}}(u\bar u + d\bar d) \quad \text{(normalised to unity)}$$

particle physics: the observed η and η' also contain $\pm s\bar s$

whereas the spin 1 multiplet is

$$\pi^+ = -u\bar d$$
$$\pi^0 = \frac{1}{\sqrt{2}}(u\bar u - d\bar d)$$
$$\pi^- = d\bar u$$

- so far we have constructed $SU(2)$ tensor reps of rank 1 (vector) and rank 2 (matrix). This can be continued as follows:

rank 3: – combine sym. tensor $\psi_{(a}\varphi_{b)}$ with χ_c

(– the combination of $\underbrace{\psi_{[a}\varphi_{b]}}$ with χ_c leads again to rank 1 as $\sim\psi^a\varphi_a$ which is an invariant. This is a peculiarity of $SU(2)$)

Clebsch-Gordan: $\overset{j}{1}\otimes\frac{1}{2} = \frac{3}{2}\oplus\frac{1}{2}$ $\left(\left(\text{or } \overset{2j+1}{3}\otimes 2 = 4 \oplus 2\right)\right)$

Yang tableau $\square\square \otimes \square = \square\square\square \oplus \begin{array}{|c|c|}\hline \, & \, \\ \hline \, \\ \cline{1-1}\end{array}$

explicitly:

$$\psi_{(a}\varphi_{b)}\chi_c = \frac{1}{3}\Big\{\left(\psi_{(a}\varphi_{b)}\chi_c + \psi_{(b}\varphi_{c)}\chi_a + \psi_{(c}\varphi_{a)}\chi_b\right) \quad \text{totally sym.}$$

$$+ \left(\psi_{(a}\varphi_{b)}\chi_c - \psi_{(b}\varphi_{c)}\chi_a\right) \quad \text{antisym in } ac \sim \varepsilon_{ac}\,\underbrace{\psi_{(d}\varphi_{b)}\chi^d}_{}$$

$$+ \left(\psi_{(a}\varphi_{b)}\chi_c - \psi_{(c}\varphi_{a)}\chi_b\right)\Big\} \quad \text{antisym in } bc \sim \varepsilon_{bc}\,\underbrace{\psi_{(d}\varphi_{a)}\chi^d}_{\equiv\eta_a}$$

- SU(3) Tensors:

the fundamental irrep of $SU(3)$ acts on 3 component spinors $\psi_{a=1,2,3}$ as

$\psi'_a = U_a{}^b\psi_b$, $U\in SU(3)$, often denoted by 3 $\square$

In contrast to $SU(2)$ complex conjugation is no longer an automorphism

$\psi'^a = U^{*a}{}_b\psi^b$ $U\in SU(3)$, denoted by $\bar{3} \not\Leftrightarrow 3$

* the ε-tensor is again an invariant (ε_{abc} tot. antisym with $\varepsilon_{123}=+1$):

$$\varepsilon'_{abc} = U_a{}^d U_b{}^e U_c{}^f \varepsilon_{def} \overset{\text{check}}{=} \overset{=1 \text{ due to } SU(3)}{\det(U)}\,\varepsilon_{abc} = \varepsilon_{abc}$$

likewise is the ε-tensor with upper indices invariant, due to $\det(U^*) = (\det U)^* = 1^* \Rightarrow \varepsilon'^{abc} = U^{*a}{}_d U^{*b}{}_e U^{*c}{}_f \varepsilon^{def} = \det U^* \varepsilon^{abc}$

* the ε-tensor can again be used to build <u>invariants</u> and to raise indices:

<u>Examples:</u> • $\psi^a = \varepsilon^{abc} \varphi_b \xi_c = \varepsilon^{abc} \frac{1}{2} \varphi_{[b} \xi_{c]}$ $\boxed{}$

transforms like a spinor with upper index as

$$\psi'^a = \varepsilon'^{abc} \varphi'_b \xi'_c = U^{*a}{}_d \underbrace{U^{*b}{}_e}_{U^{+}{}_e{}^b} \underbrace{U^{*c}{}_f}_{U^{+}{}_f{}^c} U_b{}^g U_c{}^h \varepsilon^{def} \varphi_g \xi_h \quad , \quad \underline{U^{+}{}_e{}^b U_b{}^g = \delta_e{}^g}$$

$$= U^{*a}{}_d \, \varepsilon^{dgh} \varphi_g \xi_h = U^{*a}{}_d \psi^d$$

• complete contractions like $\chi_a \psi^a$ transform like scalars and are thus invariants:

$$\chi'_a \psi'^a = U_a{}^b U^{*a}{}_c \chi_b \psi^c = \underbrace{U^{+}{}_c{}^a U_a{}^b}_{\delta_c{}^b} \chi_b \psi^c = \chi_b \psi^b$$

* note that in both cases we only used the unitarity of $U \in SU(3)$, it is not necessary for * to be an automorphism to show this.

<u>Higher dimensional irreps of SU(3):</u>

as for SU(2) we will give a few examples and draw the corresponding Young tableaux. The simplest example is as in SU(2) to combine $q\bar{q}$ (meson)

• $\psi_a \bar{\psi}^b = (\psi_a \bar{\psi}^b - \frac{1}{3} \delta_a{}^b \psi_c \bar{\psi}^c) + \frac{1}{3} \delta_a{}^b \psi_c \bar{\psi}^c$ where have split off an invariant part s.th. the remaining is traceless.

* we have 9 possible combinations of indices $a,b = 1,2,3$

and it turns out that the traceless part transforms irreducibly:

$$3 \otimes \bar{3} = 8 \oplus 1 \qquad \text{(here in dim notation), or}$$

$$\ydiagram{1} \otimes \ydiagram{1,1} = \ydiagram{2,1} \oplus \ydiagram{1,1,1}$$

The <u>Young-tableaux</u> reflect the index structure of the lower indices, so here $\bar{3}$ given by $\bar{\psi}^a = \varepsilon^{abc} T_{[bc]}$ where T is an antisymmetric tensor, rep. by $\ydiagram{1,1}$. The combination $\ydiagram{1,1,1}$ is thus totally antisym. in all lower indices (example $q=(u,d,s)$ $\Rightarrow$ $\ydiagram{1,1,1} = \eta_8 \sim u\bar{u}+d\bar{d}\ (+s\bar{s})$) and an octet ($\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta_8$).

- in order to build Baryons qqq we first need to consider the building block qq:

$$\psi_a \varphi_b = \frac{1}{2}\left(\psi_{(a}\varphi_{b)} + \psi_{[a}\varphi_{b]}\right)$$

where the antisym. part transforms like a vector with upper indices (an antisym. 3×3 matrix has 3 indep. components): thus this corresponds to

$$3 \otimes 3 = 6 \oplus \bar{3}$$

or

$$\ydiagram{1} \otimes \ydiagram{1} = \ydiagram{2} \oplus \ydiagram{1,1}$$

- combining $\psi_{(a}\varphi_{b)}$ with χ_c we can symmetrize or antisym. c with (a,b) to get $6 \otimes 3 = 10 \oplus 8$, $S_{(abc)}$ has 10 comp. (3 all equal, 6 where 2 equal, +1 where all different)
or $\ydiagram{2} \otimes \ydiagram{1} = \ydiagram{3} \oplus \ydiagram{2,1}$ (looks as for $SU(2)$, but with different dim.)
- we also may combine $\psi_{[a}\varphi_{b]}$ with χ_c which is $\bar{3} \otimes 3$ as above

$\Rightarrow$ in total $3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$

III Coordinate Transformations: the Lorentz and Poincaré Group

Notation: Lorentz group = rotations, boost, P & T, Poincaré = Lorentz + translations

in detail:

* Postulate of Homogeneity of space-Time:

→ the physics does not change: - in time & at different points in space
- when moving from 1 frame of inertia to another

→ Symmetries:

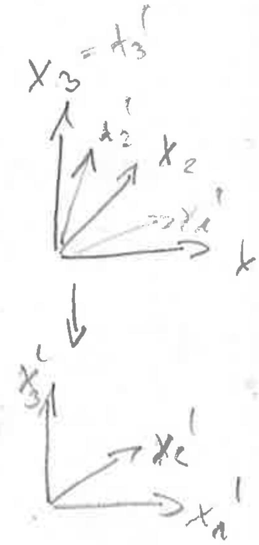
.. translation in time & translation in space | $t' = t + a_0$, $\vec{x}' = \vec{x} + \vec{a}$ or $x_\mu' = x_\mu + a_\mu;\ \mu = 0,1,2,3$; $x_0 = ct,\ (x_1, x_2, x_3) = \vec{x}$

(Noether: conservation of energy and momentum)

* rotations (see p. 32, p 44)

e.g. around x_3-axis by $\varphi \in [0, 2\pi)$ compact

$$x_0' = x_0$$
$$x_1' = x_1 \cos\varphi - x_2 \sin\varphi$$
$$x_2' = x_1 \sin\varphi + x_2 \cos\varphi$$
$$x_3' = x_3'$$



* classical mechanics - Galilei trafo

$$x_0' = x_0$$
$$x_j' = x_j - \frac{\vec{v}_j}{c} x_0$$

non-relativistic limit $c \to \infty$

* relativistic mechanics - Lorentz trafo

e.g. boost in x_3 direction

$$x_0' = \gamma\left(x_0 - x_3 \frac{v}{c}\right)$$
$$x_{1,2}' = x_{1,2}$$
$$x_3' = \gamma\left(x_3 - x_0 \frac{v}{c}\right)$$

where $\gamma(v) = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \equiv \cosh\zeta,\ \zeta \in [0, \infty)$

$\frac{v}{c} = \tanh\zeta$ non-compact

$$x_0' = x_0 \cosh\zeta - x_3 \sinh\zeta$$
$$x_1' = x_1$$
$$x_2' = x_2$$
$$x_3' = x_3 \cosh\zeta - x_0 \sinh\zeta$$

we also may have as symmetries

* time reversal T : $x_0' = -x_0$
$x_j' = +x_j$

(H-atom + $\vec{B}$-field is not T invariant)

* parity P : $x_0' = +x_0$
$x_j' = -x_j$

→ classical invariant:

the Euclidean distance between 2 points $\vec{x}$ and $\vec{y}$ is invariant

$$(\vec{x}' - \vec{y}')^2 = (\vec{x} - \vec{y})^2 = (x_j - y_j)(x_j - y_j) \equiv \|\vec{x} - \vec{y}\|^2$$ (incl. $\|\vec{x}\|$ for distance to $\vec{0}$)

under • Galilei (trivial), • rotations as these are orthogonal trafos $OO^T = \mathbb{1}$ that preserve the scalar product (and have $\det O = +1$)

• Parity P, and combination of P & rotations, this gives the second half of orthogonal trafos with $\det O = -1$ (disconnected from $\mathbb{1}$)

→ relativistic invariant:

$$(x_0' - y_0')^2 - (x_j' - y_j')(x_j' - y_j') = (x_0' - y_0')^2 - (\vec{x}' - \vec{y}')^2 \equiv c^2 \underbrace{\tau_{12}^2}_{}$$ proper time

obviously this remains invariant under rotations, P, combinations of them,
it is also invariant under T, and combination of rotat. and P

• it is also invariant under Lorentz boost

e.g. $$(x_0' - y_0')^2 - (x_j' - y_j')^2 = ((x_0 - y_0)\cosh\xi - (x_3 - y_3)\sinh\xi)^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2$$
$$- ((x_3 - y_3)\cosh\xi - (x_0 - y_0)\sinh\xi)^2$$
$$= ((x_0 - y_0)^2 - (x_3 - y_3)^2)\underbrace{(\cosh^2\xi - \sinh^2\xi)}_{=1} - (x_1 - y_1)^2 - (x_2 - y_2)^2$$ ✓

as well as combinations of boosts, P, T and rotations

• defining $\vec{u} = \frac{d\vec{x}}{dt}$ the <u>4-momentum vector</u> $p_\mu = \gamma(u)(mc, m\vec{u}) \equiv (\frac{E}{c}, \vec{p})$ of a particle with <u>mass</u> m leads to an invariant under Lorentz trafo

$$\underline{m^2c^2 = p_\mu p_\mu = \left(\frac{E}{c}\right)^2 - \vec{p}^{\,2}}$$

<u>the Lorentz group, SO(3,1) and SL(2,ℂ):</u>

the properties can be best analysed introducing

<u>the metric</u> $\underline{\eta = \mathrm{diag}(1,-1,-1,-1)}$ Minkowski ~

<u>upper indices</u> $\underline{X^\mu = \eta^{\mu\nu} X_\nu}$ $(= (x_0, -x_1, -x_2, x_3))$

<u>scalar product</u> $\underline{X^\mu Y_\mu = \eta^{\mu\nu} X_\nu Y_\mu}$, <u>norm²</u> $X^\mu X_\mu = x_0^2 - x_1^2 - x_2^2 - x_3^2 = \|X\|_4^2$

<u>Lorentz trafo</u> $X'_\mu = \Lambda_\mu{}^\nu X_\nu$, $\Lambda^\mu{}_\nu \in Gl(4, \mathbb{R})$

→ the Lorentz group are all trafos that leave the above scalar product with Minkowski metric invariant

$$X'^\mu X'_\mu = \eta^{\mu\nu} X'_\nu X'_\mu = \eta^{\mu\nu} \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta X_\alpha X_\beta \overset{!}{=} \eta^{\alpha\beta} X_\alpha X_\beta$$

$$\Leftrightarrow \boxed{(\Lambda^T \eta \Lambda)^{\alpha\beta} = \eta^{\alpha\beta}} \quad (*) \quad \Leftrightarrow \Lambda \in O(3,1)$$

with $\det(A\cdot B) = \det A \det B$ and $\det A^T = \det A$ we have

$$\Rightarrow \underline{(\det \Lambda)^2 = 1} \qquad (\text{after cancelling } \det \eta = -1 \text{ on both sides})$$

in SO(3,1) labels the number of + and − signs in the metric, S: $\det\Lambda = +1$

• the dimension of SO(3,1) is the same as O(4):

$$\dim O(4) = \frac{4(4-1)}{2} = 6 \quad = 3 \text{ generators for rotations} + 3 \text{ " for boosts}$$