• defining $\vec{u} = \frac{d\vec{x}}{dt}$ the 4-momentum vector $p_\mu = \gamma(u)(mc, m\vec{u}) \equiv (\frac{E}{c}, \vec{p})$ of a particle with mass m leads to an invariant under Lorentz trafo

$$m^2c^2 = p_\mu p_\mu = \left(\frac{E}{c}\right)^2 - \vec{p}^{\,2}$$

the Lorentz group, SO(3,1) and SL(2,C):

the properties can be best analysed introducing

the metric $\eta = \mathrm{diag}(1,-1,-1,-1)$ Minkowski "

upper indices $x^\mu = \eta^{\mu\nu} x_\nu$ $(= (x_0, -x_1, -x_2, x_3))$

scalar product $x^\mu y_\mu = \eta^{\mu\nu} x_\nu y_\mu$, norm² $x^\mu x_\mu = x_0^2 - x_1^2 - x_2^2 - x_3^2 = \|x\|_4^2$

Lorentz trafo $x'_\mu = \Lambda_\mu{}^\nu x_\nu$, $\Lambda^\mu{}_\nu \in GL(4,\mathbb{R})$

→ the Lorentz group are all trafos that leave the above scalar product with Minkowski metric invariant

$$x'^\mu x'_\mu = \eta^{\mu\nu} x'_\nu x'_\mu = \eta^{\mu\nu} \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta x_\alpha x_\beta \overset{!}{=} \eta^{\alpha\beta} x_\alpha x_\beta$$

$$\Leftrightarrow \boxed{(\Lambda^T \eta \Lambda)^{\alpha\beta} = \eta^{\alpha\beta}} \quad (*) \quad \Leftrightarrow \Lambda \in O(3,1)$$

with $\det(A \cdot B) = \det A \det B$ and $\det A^T = \det A$ we have

$$\Rightarrow \boxed{(\det\Lambda)^2 = 1}$$ (after cancelling $\det\eta = -1$ on both sides)

in SO(3,1) labels the number of + and − signs in the metric, S: $\det\Lambda = +1$

• the dimension of SO(3,1) is the same as O(4):

$$\dim O(4) = \frac{4(4-1)}{2} = 6$$ = 3 generators for rotations + 3 " for boosts

$\Lambda = I = \mathrm{diag}(1,1,1,1)$ identity

• note that $\Lambda = I_P = \mathrm{diag}(1,-1,-1,-1)$ Parity $(= z)$

$\Lambda = I_T = \mathrm{diag}(-1,+1,+1,+1)$ time reversal

$\Lambda = I_{PT} = \mathrm{diag}(-1,-1,-1,-1)$ P & T

are metric preserving Lorentz trafos!

Topology of the Lorentz-group:

from ⊛ at $\alpha = \beta = 0$ we have

$$\eta^{00} = 1 = (\Lambda^T \eta \Lambda)^{00} = \eta^{\mu\nu} \Lambda_\mu{}^0 \Lambda_\nu{}^0 = (\Lambda_0{}^0)^2 - \sum_{j=1}^{3} (\Lambda_j{}^0)^2 = 1$$

$\Rightarrow$ $(\Lambda_0{}^0)^2 \geq 1$ which has 2 possible solutions:

orthochrone Lorentz trafo $\boxed{\Lambda_0{}^0 \geq 1}$ $\mathcal{L}^\uparrow$

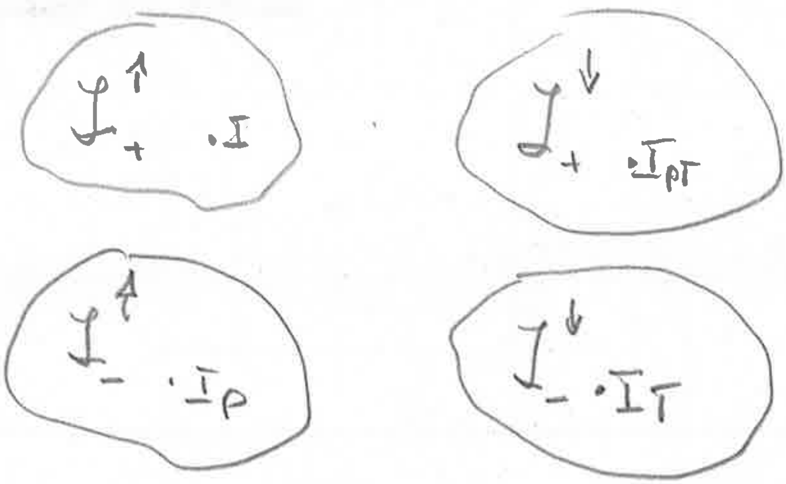
antichrone " " $\boxed{\Lambda_0{}^0 \leq -1}$ $\mathcal{L}^\downarrow$

• defining with $\det \Lambda = +1$ proper Lorentz-trafos $\mathcal{L}_+$: $SO(3,1)$

and -1 $\mathcal{L}_-$

we have 4 possibilities (instead of 2 for $O(2,1)$):

$$\mathcal{L} = \mathcal{L}_+^\uparrow \cup \mathcal{L}_-^\uparrow \cup \mathcal{L}_+^\downarrow \cup \mathcal{L}_-^\downarrow$$ which are topologically disjoint



* we will now show that $\mathcal{L}_+^\uparrow \cong SL(2,\mathbb{C})$, and then classify its irreps.

(the relation $SO(3,1) = \mathcal{L}_+^\uparrow \cup \mathcal{L}_+^\downarrow$ to $SL(2,\mathbb{C})$ is reminiscent to $SU(2)/_{\mathbb{Z}_2} \cong SO(3)$)

<u>Homomorphism to $SL(2,\mathbb{C})$</u>:

recall that $GL(n,\mathbb{C})$ are invertible $n\times n$ matrices with complex elements and that $SL(n,\mathbb{C}) \subset GL(n,\mathbb{C})$ has determinant 1

1) we <u>represent</u> any <u>4-vector x^μ</u> as a <u>2×2 Hermitian matrix</u>, using the following <u>bijection</u>:

$\sigma_0 = \begin{pmatrix}1&0\\0&1\end{pmatrix}$ and the Pauli matrices $\sigma_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $\sigma_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

form a basis for any complex Hermitian 2×2 matrix $A = A^\dagger$.

$x^\mu \to A$: $A = x^\mu \sigma_\mu = \begin{pmatrix} x^0+x^3 & x^1-ix^2 \\ x^1+ix^2 & x^0-x^3 \end{pmatrix}$ with $x^{0,1,2,3} \in \mathbb{R}^4$

$\left(= \begin{pmatrix} a & b \\ b^* & c\end{pmatrix},\ a,c \in \mathbb{R},\ b\in\mathbb{C} \right)$.

The map from $\boxed{H_2 \equiv \{A_{ij} \in \mathbb{C},\ i,j=1,2,\ A = A^\dagger\}} \overset{1-1}{\longleftrightarrow} \mathbb{R}^4$ is bijective:

defining $\tilde{\sigma}^\mu = (\sigma_0, \vec{\sigma}_i)$ we can map back using

$A \to x^\mu$: $\underline{x^\mu \overset{!}{=} \frac{1}{2}\mathrm{Tr}(\tilde{\sigma}^\mu A)} = \frac{1}{2}\mathrm{Tr}(\tilde{\sigma}^\mu x^\nu \sigma_\nu) = x^\nu \overbrace{\tfrac{1}{2}\mathrm{Tr}(\tilde{\sigma}^\mu \sigma_\nu)}^{\delta^\mu{}_\nu} = x^\mu$

as all σ_i are Traceless and $\sigma_j\sigma_i = i\epsilon_{jik}\sigma_k$

* it also holds

$$\det A = (x^0+x^3)(x^0-x^3) - (x^1-ix^2)(x^1+ix^2) = x^{0\,2} - x^{i\,2} = x_0^2 - x_i^2$$
$$= x^\mu x_\mu$$

---

2) We will now construct for any $M \in SL(2,\mathbb{C})$ the corresp. $\Lambda(M)$ and show $\Lambda(M) \in \mathcal{L}_+^\uparrow$:

Let $M \in SL(2,\mathbb{C})$: $\det M = 1$, and let $A \in H_2$
($\exists$ corresp. x^μ)

$\Rightarrow$ the conjugation $\underline{A' \equiv MAM^+}$ $\in H_2$ as $(A')^+ = M^{++}A^+M^+ = MAM^+$

• This conjugation is linear: $(\alpha A + \beta B)' = M(\alpha A + \beta B)M^+ = \alpha A' + \beta B'$
and remains in H_2 for $\alpha, \beta \in \mathbb{R}$, $A, B \in H_2$

we define $x'^\mu \equiv \frac{1}{2}\operatorname{Tr}(\tilde{\sigma}^\mu A')$, with $x^\mu = \frac{1}{2}\operatorname{Tr}(\tilde{\sigma}^\mu A)$

and write $x'^\mu = [\Lambda(M)]^\mu{}_\nu x^\nu$ for the induced linear matrix rep.

It remains to be shown that this is a Lorentz trafo $\in \mathcal{L}_+^\uparrow$:

• $x'^\mu x'_\mu = \det(MAM^+) = |\det M|^2 \det A = 1 \cdot \det A = x^\mu x_\mu$

$\Rightarrow$ $\Lambda(M)$ preserves the norm of x^μ and

• $x'^\mu y'_\mu = \frac{1}{2}(x'^\mu y'_\mu + y'^\mu x'_\mu) = \frac{1}{2}\left((x'+y')^\mu (x'+y')_\mu - x'^\mu x'_\mu - y'^\mu y'_\mu\right)$

$$= x^\mu y_\mu \qquad \overset{\shortparallel}{(x+y)^\mu (x+y)_\mu} \qquad \overset{\shortparallel}{x^\mu x_\mu} \qquad \overset{\shortparallel}{y^\mu y_\mu}$$

$\Rightarrow$ $\Lambda(M)$ preserves the scalar product of any 2 vectors!

Q: How to determine $\Lambda(M)^\mu{}_\nu$ explicitly?

$$x'^\mu = \frac{1}{2}\operatorname{Tr}(\tilde{\sigma}^\mu MAM^+) = \frac{1}{2}\operatorname{Tr}(\tilde{\sigma}^\mu M \underset{\uparrow \in \mathbb{R}^4}{x^\nu} \sigma_\nu M^+) = \boxed{\underbrace{\frac{1}{2}\operatorname{Tr}(\tilde{\sigma}^\mu M \sigma_\nu M^+)}_{\equiv [\Lambda(M)]^\mu{}_\nu} x^\nu}$$

in particular:

$[\Lambda(M)]^0{}_0 = \frac{1}{2}\operatorname{Tr}(MM^+) = \frac{1}{2} M_{ij} M^*_{ji} \geq 0$ so orthochrone $\uparrow$

• it holds $[\Lambda(M = \mathbb{1})]^\mu{}_\nu = \frac{1}{2}\operatorname{Tr}(\tilde{\sigma}^\mu \sigma_\nu) = \delta^\mu{}_\nu$ which has $\det \Lambda(\mathbb{1}) = +1$
which picks the proper Lorentz trafo, so $\mathcal{L}_+^\uparrow$.
Since $SL(2,\mathbb{C})$ is simply connected, $\det M \neq 0$ is a continuous function of M, $\det[\Lambda(M)] = +1$ for all M's $\in SL(2,\mathbb{C})$

3) It remains to show that $SL(2,\mathbb{C}) \to L_+^\uparrow$, $f: M \to \Lambda(M)$

is a <u>group homomorphism</u>,

that is $f(ab) = f(a)\cdot f(b)$:

for $A'' = MM'A(MM')^+ = M(\underbrace{M'AM'^+}_{A'=})M^+$ we have

$A'' = x''^\mu \sigma_\mu = \underline{\Lambda(MM')^\mu{}_\nu x^\nu \sigma_\mu}$

and $= MM'(x^\nu\sigma_\nu)(MM')^+ = M(M'AM'^+)M^+ = M(\underbrace{\Lambda(M')^\nu{}_\rho x^\rho \sigma_\nu}_{= x'^\nu \sigma_\nu})M^+$

$= \Lambda(M)^\mu{}_\nu x'^\nu \sigma_\mu = \underline{\Lambda(M)^\mu{}_\nu \Lambda(M')^\nu{}_\rho x^\rho \sigma_\mu}$ which is what we claimed.

<u>Generators of $SO(3,1)$</u>

the groups $O(3,1)$ and $SO(3,1)$ are non-compact, as p115

$\zeta = \frac{1}{2}\ln\left(\frac{1+\frac{v}{c}}{1-\frac{v}{c}}\right) = \operatorname{artanh}\frac{v}{c} \in [0,\infty)$, ζ <u>rapidity</u>

- still we can define its generators and compute its algebra

<u>Rotations</u>: trivial extension of p74 by 4th columns, e.g.

write p115 as $x'_\mu = \begin{pmatrix} 1 & 0\,0\,0 \\ 0 & \\ 0 & R_3(\varphi) \\ 0 & \end{pmatrix}_\mu{}^\nu x_\nu = \begin{pmatrix} 1&0&0&0 \\ 0&c&-s&0 \\ 0&s&c&0 \\ 0&0&0&1 \end{pmatrix}_\mu{}^\nu x_\nu$ $c = \cos\varphi$, $s = \sin\varphi$

and generator $-i\bar{X}_3 \equiv \left.\frac{dR_3}{d\varphi}\right|_{\varphi=0} = \begin{pmatrix} 0&0&0&0 \\ 0&0&-1&0 \\ 0&1&0&0 \\ 0&0&0&0 \end{pmatrix}$, (we called $-X_3 = T_3$ before)

$\Leftrightarrow \bar{X}_3 = \begin{pmatrix} 0&0&0&0 \\ 0&0&-i&0 \\ 0&i&0&0 \\ 0&0&0&0 \end{pmatrix}$, ditto $\bar{X}_2 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&+i \\ 0&0&0&0 \\ 0&-i&0&0 \end{pmatrix}$, $\bar{X}_1 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&-i \\ 0&0&+i&0 \end{pmatrix}$

with algebra $\underline{[\bar{X}_a, \bar{X}_b] = i\varepsilon_{ab}{}^c \bar{X}_c}$ $a,b,c = 1,2,3$ (p. 74)

with matrix rep $\left((\bar{X}_j)_k{}^l = -i\varepsilon_{jk}{}^l \text{ as } 3\times3 \Rightarrow\right)(\bar{X}_j)_\mu{}^\nu = -i\varepsilon_{j\mu}{}^\nu = -i\varepsilon_{j\mu\rho}\eta^{\rho\nu}$ $=0$ if μ or $\rho = 0$