<u>Lorentz boost</u> : we had

$$x' = \begin{pmatrix} ch & 0 & 0 & -sh \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -sh & 0 & 0 & ch \end{pmatrix} x \quad , \quad ch = \cosh(\varsigma),\ sh = \sinh(\varsigma)$$
$$\qquad\qquad \equiv B_3(\varsigma) \qquad\qquad\qquad\qquad \in [0,\infty)$$

so def $\quad Y_3 = i \frac{dB_3}{d\varsigma}\Big|_{\varsigma=0} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$

ditto $\quad Y_1 = i \frac{dB_1}{d\varsigma}\Big|_{\varsigma=0} = i \frac{d}{d\varsigma}\Big| \begin{pmatrix} ch & -sh & 0 & 0 \\ -sh & ch & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

which can be written as for its matrix elements $\quad \underline{(Y_k)_\mu{}^\nu = -i(\eta_{\mu 0}\delta_k^\nu - \eta_{\mu k}\delta_0^\nu)}, \; Y_k^\dagger = -Y_k$

<u>claim:</u> this leads to the following algebra

$$\boxed{\begin{aligned} [X_a, X_b] &= i\varepsilon_{ab}{}^c X_c \\ [X_a, Y_b] &= i\varepsilon_{ab}{}^c Y_c \\ [Y_a, Y_b] &= -i\varepsilon_{ab}{}^c X_c \end{aligned}}$$

which is the SO(3) subalg

almost as SO(4) (see ch 10.2)

check: • $[X_a, Y_b] = (-i)^2 \varepsilon_{a\mu}{}^\rho (\eta_{\rho 0}\delta_b^\nu - \eta_{\rho b}\delta_0^\nu) - (-i)^2 \varepsilon_{a\rho}{}^\nu (\eta_{\mu 0}\delta_b^\rho - \eta_{\mu b}\delta_0^\rho)$

$= + \varepsilon_{a\mu b}\delta_0^\nu + \varepsilon_{ab}{}^\nu \eta_{\mu 0} = i(-i)\varepsilon_{ab}{}^c(\eta_{\mu 0}\delta_c^\nu - \eta_{\mu c}\delta_0^\nu)$

• $[Y_a, Y_b] = (-i)^2 (\eta_{\mu 0}\delta_a^\rho - \eta_{\mu a}\delta_0^\rho)(\eta_{\rho 0}\delta_b^\nu - \eta_{\rho b}\delta_0^\nu) - (a \leftrightarrow b)$

$\eta_{a0} = 0$

$= (-1)(-\eta_{\mu 0}\eta_{ab}\delta_0^\nu - \eta_{\mu a}\eta_{00}\delta_b^\nu + \eta_{\mu 0}\eta_{ba}\delta_0^\nu + \eta_{\mu b}\eta_{00}\delta_a^\nu) = -i\varepsilon_{ab}{}^c (X_c)_\mu{}^\nu$

$= \eta_{\mu a}\delta_b^\nu - \eta_{\mu b}\delta_a^\nu$

as $(-i)\epsilon_{ab}{}^{c}(\bar{X}_c)_\mu{}^\nu = (-i)^2 \epsilon_{ab}{}^{c}\epsilon_{c\mu\rho}\,\eta^{\rho\nu}$

$$= (-1)\left(\delta_{a\mu}\delta_{b\rho} - \delta_{a\rho}\delta_{b\mu}\right)\eta^{\rho\nu}$$

$$= (-1)\left(\eta_{a\mu}\eta_{b\rho} - \eta_{a\rho}\eta_{b\mu}\right)\eta^{\rho\nu}$$

$$= \eta_{\mu a}\delta_b{}^\nu - \eta_{\mu b}\delta_a{}^\nu$$

• the algebra of the Lorentz group can be seen to decouple into two indep. $SU(2)$'s, with the following field redef.

$$X_a^{(\pm)} \equiv \frac{1}{2}\left(\mathcal{X}_a \pm i Y_a\right)$$ leading to

$$\boxed{\begin{aligned} [X_a^{(+)}, X_b^{(+)}] &= i\varepsilon_{abc} X_c^{(+)} \\ [X_a^{(-)}, X_b^{(-)}] &= i\varepsilon_{abc} X_c^{(-)} \\ [X_a^{(+)}, X_c^{(-)}] &= 0 \end{aligned}}$$

* notice that we are considering a <u>complex extension</u> of the Lie-algebra here, due to $\pm i Y_a$:

we had $\mathcal{X}_a^{+} = +\mathcal{X}_a$ <u>Hermitian generators</u>

but $Y_a^{+} = -Y_a$ <u>anti-Hermitian</u>

so the $\mathcal{X}_a^{(\pm)}$ are Hermitian.

This is why <u>$\mathcal{L}_+^{\uparrow} \cong Sl(2,\mathbb{C})$</u> (rather than $SU(2)\times SU(2)$ as for $SO(4)$)

$\Rightarrow$ we can write a Lorentz trafo Λ in the component of $\mathcal{L}_+^{\uparrow} \ni \Lambda$ by

<u>$\Lambda = \exp[i\theta_a \mathcal{X}_a + i\zeta_a Y_a]$</u> : we obtain a

non-unitary representation. This is because the Lorentz group is <u>non-compact</u>.

Irreducible Representations of $SO(3,1)$

- these can be now labelled by the spins of the 2 $SU(2)$'s (j_1, j_2) with $(2j_1+1)(2j_2+1)$ degrees of freedom

Examples: 1. the lowest nontrivial ($\neq (0,0)$) irrep is the 2-dimensional $(\frac{1}{2}, 0)$ (or $(0, \frac{1}{2})$) acting on 2-spinors.

In the standard model of elementary particle physics the massless neutrinos form such a $(\frac{1}{2}, 0)$ irrep

2. the electron represented by a Dirac or 4 spinor is a reducible rep $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$. The reason is that the Dirac eq. has the anti particle of the electron, the positron, as a solution, in contains Parity $P = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

$\Rightarrow$ conjugation under parity $\underline{X} \to P \underline{X} P^{-1} = + \underline{X}$

but $\quad Y \to P Y P^{-1} = - Y$

which leads to $X^{(\pm)} \to P X^{(\pm)} P^{-1} = X^{(\mp)}$, so both $X^{(\pm)}$ are contained in this rep in a symmetric manner

3. the irrep $(\frac{1}{2}, \frac{1}{2})$ with 4 d.o.f. corresponds to Lorentz 4 vectors e.g. A_μ
4. the reducible rep $(1,0) \oplus (0,1)$ has 6 d.o.f. and is carried by anti-symmetric tensors $F_{\mu\nu} = -F_{\nu\mu}$, e.g. by the electromagnetic field strength tensor with indep. components $E_j = -F_{0j}$, $B_j = -\varepsilon_{jkl} F_{kl}$

RHS $(-i)i\left[\eta_{\mu\rho}(x_\nu\partial_\sigma - x_\sigma\partial_\nu) - \eta_{\mu\sigma}(x_\nu\partial_\rho - x_\rho\partial_\nu) + \eta_{\nu\sigma}(x_\mu\partial_\rho - x_\rho\partial_\mu)\right.$

$\left. - \eta_{\nu\rho}(x_\mu\partial_\sigma - x_\sigma\partial_\mu)\right]$

LHS $-(x_\mu\partial_\nu - x_\nu\partial_\mu)(x_\rho\partial_\sigma - x_\sigma\partial_\rho)$

$+ (x_\rho\partial_\sigma - x_\sigma\partial_\rho)(x_\mu\partial_\nu - x_\nu\partial_\mu)$

$= -x_\mu\eta_{\nu\rho}\partial_\sigma + x_\mu\eta_{\nu\sigma}\partial_\rho + x_\nu\eta_{\mu\rho}\partial_\sigma - x_\nu\eta_{\mu\sigma}\partial_\rho$

$+ x_\rho\eta_{\sigma\mu}\partial_\nu - x_\rho\eta_{\sigma\nu}\partial_\mu - x_\sigma\eta_{\rho\mu}\partial_\nu + x_\sigma\eta_{\rho\nu}\partial_\mu$

The Poincaré Group

Because ~~ = Lorentz + translation

$$x'_\mu = \Lambda_\mu{}^\nu x_\nu + a_\mu$$ does not act linearly

due to the inhomogeneous term we cannot find a matrix rep.

$\rightarrow$ rep. on functions $f(x_0, x_1, x_2, x_3)$.

The generators obtained by $+i \frac{\partial f}{\partial \alpha}\big|_{\alpha=0}$ will then give us the algebra.

• define $\partial_\mu = \left(\frac{\partial}{\partial x_0}, -\frac{\partial}{\partial x_1}, -\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_3}\right)$ as this transforms

as x_μ under Lorentz trafos:

e.g. $\partial_\mu (\underbrace{x^\beta x_\beta}_{\text{inv.}}) = \partial_\mu (x_0^2 - x_j x_j) = 2(x_0, x_1, x_2, x_3) = 2x_\mu$

$\Rightarrow \partial^\mu = \eta^{\mu\nu}\partial_\nu$

$\Rightarrow \boxed{P_\mu = i\partial_\mu}$ generates infinitesimal translations:

$f(x_\mu + a_\mu) = f(x_\mu) + a_\mu \partial^\mu f(x_\nu) + O(a^2) \approx \exp[-i a^\beta P_\beta]$

$\Rightarrow \quad P_\mu = +i \frac{\partial f}{\partial a^\mu}\big|_{a^\mu=0}$ as we already saw for $\mu = 1,2,3$ in the beginning on p. 6

• we also had $L_{ke} = i(x_k \partial_e - x_e \partial_k)$ $(\sim M_{ke})$ as generator for rotations there, and the corresponding algebra.

We will now derive the following Poincaré algebra

$$[P_\mu, P_\nu] = 0$$
$$[P_\mu, L_{\beta\delta}] = i(\eta_{\mu\beta} P_\delta - \eta_{\mu\delta} P_\beta)$$
$$[L_{\mu\nu}, L_{\beta\delta}] = -i(\eta_{\mu\beta} L_{\nu\delta} - \eta_{\mu\delta} L_{\nu\beta} + \eta_{\nu\delta} L_{\mu\beta} - \eta_{\nu\beta} L_{\mu\delta})$$

with $\boxed{L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)}$ and relate that to our previous rep of Lorentz

Lorentz generators

note that we expand $\Lambda \in \mathcal{L}_+^\uparrow$ around the identity, so infinitesimally

$$x_\mu' = (\delta_\mu^{\ \nu} + \alpha_\mu^{\ \nu})\, x_\nu = \Lambda_\mu^{\ \nu} x_\nu$$

$\overset{\eta_{\nu\rho}}{\Longleftrightarrow}$

$$\underline{\Lambda_{\mu\rho} = \eta_{\mu\rho} + \alpha_{\mu\rho}} \qquad \text{where} \quad \alpha_{\mu\rho} \overset{!}{=} -\alpha_{\rho\mu} \quad \circledast$$

has 6 indep. components ✓

as we had:

$$\eta_{\rho\nu} = (\Lambda^T \eta \Lambda)_{\rho\nu} = \Lambda^T{}_\rho{}^\sigma \eta_{\sigma\mu} \Lambda^\mu{}_\nu = \Lambda_{\mu\rho} \Lambda^\mu{}_\nu$$

$$= (\eta_{\mu\rho} + \alpha_{\mu\rho})(\delta^\mu{}_\nu + \alpha^\mu{}_\nu) = \eta_{\nu\rho} + \alpha_{\nu\rho} + \alpha_{\rho\nu} + \mathcal{O}(\varepsilon^2)$$

- we will show now that such infinitesimal trafo can indeed be generated on polynomial functions (here on x_μ itself) with the above def. $L_{\mu\nu}$:

$$\underline{x_\mu'} = x_\mu + \alpha_\mu^{\ \nu} x_\nu \overset{!}{=} \underline{\exp[\ \ i\tfrac{1}{2} \alpha_{\rho\sigma} L^{\rho\sigma}]\, x_\mu}$$

$$= x_\mu - \tfrac{1}{2} \alpha_{\rho\sigma} (x^\rho \partial^\sigma - x^\sigma \partial^\rho) x_\mu + \mathcal{O}(\alpha^2) = x_\mu - \tfrac{1}{2}\alpha_{\rho\sigma}(x^\rho \delta_\mu^\sigma - x^\sigma \delta_\mu^\rho) + \mathcal{O}(\alpha^2)$$

$$= x_\mu + \alpha_\mu^{\ \rho} x_\rho \quad \text{using the antisymmetry}$$

- checking the actual commutation relations $[L_{\mu\nu}, L_{\rho\sigma}]$ is left as an exercise, using $\partial_\mu x^\rho = \delta_\mu^{\ \rho}$ and $\partial_\mu x_\rho = \partial_\mu x^\nu \eta_{\nu\rho} = \eta_{\mu\rho}$,

and we have (ex12) $[P_\mu, L_{\rho\sigma}] = i^2 \partial_\mu (x_\rho \partial_\sigma - x_\sigma \partial_\rho) - i^2 (x_\rho \partial_\sigma - x_\sigma \partial_\rho)\partial_\mu$

$$= -\eta_{\mu\rho}\partial_\sigma + \eta_{\mu\sigma}\partial_\rho = i(\eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho) \checkmark$$

* can we identify the $SU(2)$ generators of the Lorentz subalgebra in the previous setting? (p. 122)

Yes: in defining $\boxed{X_a = \frac{1}{2}\varepsilon_{abc} L_{bc}} \Leftrightarrow \underline{\varepsilon_{adf} X_a} = \frac{1}{2}(\delta_{bd}\delta_{cf} - \delta_{bf}\delta_{cd}) L_{bc} = \underline{L_{df}}$

we have $\underline{[X_a, X_d]} = \frac{1}{4}\varepsilon_{abc}\varepsilon_{def}[L_{bc}, L_{ef}]$, with $\eta_{ke} = -\delta_{ke}$

$$= \frac{+i}{4}\varepsilon_{abc}\varepsilon_{def}(\delta_{be}L_{cf} - \delta_{bf}L_{ce} + \delta_{cf}L_{be} - \delta_{ce}L_{bf})$$

$$= i\varepsilon_{ace}\varepsilon_{dfe}L_{cf} = i(\delta_{ad}\cancel{\delta_{cf}} - \delta_{af}\delta_{cd})L_{cf} = iL_{ad} = \underline{i\varepsilon_{adc}X_c}$$

and $\underline{Y_a = L_{0a}}$ yields the commutation relations on p. 122

(check, e.g. $[X_a, Y_d] = \frac{1}{2}\varepsilon_{abc}[L_{bc}, L_{0d}]$

$$= \frac{-i}{2}\varepsilon_{abc}(\cancel{\eta_{b0}L_{cd}} - \eta_{bd}L_{c0} + \eta_{cd}L_{b0} - \cancel{\eta_{c0}L_{bd}})$$

$$= \frac{-i}{2}\varepsilon_{adc}L_{c0} + \frac{i}{2}\varepsilon_{abd}L_{b0} = i\varepsilon_{adb}L_{0b} = i\varepsilon_{adb}Y_b)$$

etc.

$\Rightarrow$ we get the following Poincaré algebra on $\{P_\mu, X_a, Y_b\}$

$$\underline{[P_\mu, P_\nu] = 0}$$

- P_0 is a scalar under rotations: $[P_0, X_a] = \frac{1}{2}\varepsilon_{abc}[P_0, L_{bc}] = \frac{i}{2}\varepsilon_{abc}(\cancel{\eta_{0b}P_c - \eta_{0c}P_b}) = \underline{0}$

- P_d is a vector under rotations: $\underline{[P_d, X_a]} = \frac{1}{2}\varepsilon_{abc}[P_d, L_{bc}] = \frac{i}{2}\varepsilon_{abc}(\eta_{db}P_c - \eta_{dc}P_b) = \underline{i\varepsilon_{dac}P_c}$

- only P_0 and P_a are affected under boost Y_a:

$$\underline{[P_0, Y_a]} = [P_0, L_{0a}] = i(\eta_{00}P_a - \cancel{\eta_{0a}P_0}) = \underline{iP_a}$$

$$\underline{[P_d, Y_a]} = [P_d, L_{0a}] = i(\eta_{d0}P_a - \eta_{da}P_0) = \underline{i\delta_{ad}P_0}$$

$\Rightarrow [P_0, X_a^{(\pm)}] = \mp P_a$, $[P_d, X_a^{(\pm)}] = i\varepsilon_{dac}P_c \mp \delta_{da}P_0$, both $\neq 0$

$X_a^{(\pm)} = X_a \pm iY_a$

cannot use P_μ (3,1,1) to label reps