Yes: in defining $\boxed{X_a = \frac{1}{2}\varepsilon_{abc} L_{bc}}$ <=> $\varepsilon_{adf} X_a = \frac{1}{2}(\delta_{bd}\delta_{cf} - \delta_{bf}\delta_{cd}) L_{bc} = L_{df}$

we have $[X_a, X_d] = \frac{1}{4}\varepsilon_{abc}\varepsilon_{def}[L_{bc}, L_{ef}]$, with $\eta_{ke} = -\delta_{ke}$

$= \frac{+i}{4}\varepsilon_{abc}\varepsilon_{def}(\delta_{be}L_{cf} - \delta_{bf}L_{ce} + \delta_{cf}L_{be} - \delta_{ce}L_{bf})$

$= i\varepsilon_{ace}\varepsilon_{dfe}L_{cf} = i(\delta_{ad}\cancel{\delta_{cf}} - \delta_{af}\delta_{cd})L_{cf} = iL_{ad} = i\varepsilon_{adc}X_c$

and $Y_a = L_{0a}$ yields the commutation relations on p. 122

(check, e.g. $[X_a, Y_d] = \frac{1}{2}\varepsilon_{abc}[L_{bc}, L_{0d}]$

$= \frac{-i}{2}\varepsilon_{abc}(\cancel{\eta_{b0}L_{cd}} - \eta_{bd}L_{c0} + \eta_{cd}L_{b0} - \cancel{\eta_{c0}L_{bd}})$

$= \frac{-i}{2}\varepsilon_{adc}L_{c0} + \frac{i}{2}\varepsilon_{abd}L_{b0} = i\varepsilon_{adb}L_{0b} = i\varepsilon_{adb}Y_b$)

etc.

=> we get the following Poincaré algebra on $\{P_\mu, X_a, Y_b\}$

- $[P_\mu, P_\nu] = 0$
- P_0 is a scalar under rotations: $[P_0, X_a] = \frac{1}{2}\varepsilon_{abc}[P_0, L_{bc}] = \frac{i}{2}\varepsilon_{abc}(\eta_{0b}P_c - \eta_{0c}P_b) = 0$
- P_d is a vector under rotations: $[P_d, X_a] = \frac{1}{2}\varepsilon_{abc}[P_d, L_{bc}] = \frac{i}{2}\varepsilon_{abc}(\eta_{db}P_c - \eta_{dc}P_b) = i\varepsilon_{dac}P_c$
- only P_0 and P_a are affected under boost Y_a:

$[P_0, Y_a] = [P_0, L_{0a}] = i(\eta_{00}P_a - \eta_{0a}P_0) = iP_a$

$[P_d, Y_a] = [P_d, L_{0a}] = i(\eta_{d0}P_a - \eta_{da}P_0) = i\delta_{ad}P_0$

=> $[P_0, X_a^{(\pm)}] = \mp P_a$, $[P_d, X_a^{(\pm)}] = i\varepsilon_{dac}P_c \mp \delta_{da}P_0$, cannot use P_μ (S1(i)) to label rep's, both $\neq 0$

$X_a^{(\pm)} = X_a \pm iY_a$

Representations of the Poincaré group

How can we label such reps? For SU(2) (spin) we used the eigenvalues of the operator $J_i J_i = \vec{J}^2$ ($t_j(j+1)$) which commutes with all elements of the algebra $[\vec{J}^2, J_i] = 0$ (for Lorentz 2 SU(2)'s X_a, Y_a)

Such an operator is called quadratic Casimir operator (see also p. 82)

* for Poincaré there are 2 such operators:

- $P^2 = P^\mu P_\mu$ with eigenvalue $p^\mu p_\mu = c^2 m^2$ (set $c=1$ now)

as $$\boxed{[P^2, P_\mu] = 0 = [P^2, L_{\rho\sigma}]} \quad (\text{ex } 12)$$

$\Rightarrow$ we will distinguish reps with $m \neq 0$ and $m = 0$ below

- $$\boxed{W_\mu \equiv -\frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} L^{\nu\rho} P^\sigma}$$ Pauli-Lubanski vector

where $\varepsilon_{0123} = +1$ is the totally antisym. ε-tensor in 4 dimensions,

and $W^2 = W^\mu W_\mu$ is another such operator

$$\boxed{[W^2, P_\mu] = 0 = [W^2, L_{\rho\sigma}]}$$

- W_μ transforms as a 4-vector and it is orthogonal to P_μ with

$$W_\mu P^\mu = -\frac{1}{2} \underbrace{\varepsilon_{\mu\nu\rho\sigma}}_{\text{antisym}} L^{\nu\rho} \underbrace{P^\sigma P^\mu}_{\text{sym}} = 0$$

Massive representations: $m \neq 0$

- for $m \neq 0$ we can go to the local rest frame with $\bar{p}_\mu = (m, \vec{0})$

$$\Rightarrow W_0 = -\frac{1}{2} \varepsilon_{0\nu\rho\sigma} L^{\nu\rho} \underbrace{P^\sigma}_{\neq 0 \text{ for } \sigma = 0} \quad \text{gives } 0$$

$$W_i = -\frac{1}{2}\varepsilon_{i\nu\rho\sigma} L^{\nu\rho} P^\sigma = +\frac{1}{2}\varepsilon_{0ijk} L^{jk} P^0$$

$\Rightarrow$ acts like $\frac{1}{2}\varepsilon_{ijk} L^{jk} m = m \Sigma_i$ = angular momentum on states in rest frame and

$$W^2 \to -W_i^2 = -m^2 \Sigma_i \Sigma_i \quad \text{acts as ang. mom. squared (casimir)}$$

$\Rightarrow$ we can use m and s as labels:

$$P^2 |m, s, \ldots\rangle = m^2 |m, s, \ldots\rangle$$
$$W^2 |m, s, \ldots\rangle = -m^2 s(s+1) |m, s, \ldots\rangle$$

* because P_μ commutes with P^2 and W^2 we can consider an arbitrary frame, with the same quantum numbers.

* What else is there?

due to $m^2 = E^2 - \vec{p}^2$ we may specify $\vec{p}$ of a state

Can we also use Σ_a (like J_z-comp)? NO, $[\Sigma_a, P_\mu] \neq 0$

$\rightsquigarrow$ use Helicity $\Sigma_a P_a$ with eigenvalue $\lambda |\vec{p}|$ (still def. for $\vec{p}=0$)

as $[\Sigma_a P_a, P_\mu] = 0$

$\Rightarrow$ the infinite-dim vector space rep. of Poincaré for $m \neq 0$ has the following states $|m, s; \vec{p}, \lambda\rangle$ with

$$P_\mu |m, s; \vec{p}, \lambda\rangle = p_\mu |m, s; \vec{p}, \lambda\rangle$$
$$\Sigma_a P_a | \quad\quad \rangle = \lambda |\vec{p}| | \quad\quad \rangle$$

action of the Poincaré group on Fock vector space:

translations: $x_\mu \to x_\mu + a_\mu$:

$$|m, s; \vec{p}\, \lambda\rangle \to e^{-i p^\mu a_\mu} |m, s; \vec{p}, \lambda\rangle$$

Lorentz trafos:

* first rep $|m, s; \vec{p}, \lambda\rangle$ by trafo acting on rest frame $\bar{p}$
  let us choose $\lambda = s_3$ ev of $\bar{X}_3$ in the rest frame

$$\Rightarrow \quad |m, s; \vec{p}, \lambda\rangle = \Lambda_p |m, s; \vec{0}, s_3\rangle \equiv B_p R(\vec{p}) |m, s; \vec{0}, \lambda\rangle$$

with Λ_p = rotation of 3-axis into $\vec{p}$ $R(\vec{p})$ and then boost

$$B_p = \exp\Big(-i \underbrace{\tfrac{\vec{p}_a}{m}}_{\hat{=} \, \mathcal{I}_a} Y_a\Big) \text{ in } \vec{p}\text{-direction}$$

* how boost $p \to p'$ described by $\Lambda_{p' \leftarrow p}$ on $|m, s; \vec{p}\, \lambda\rangle$

$$\Lambda_{p' \leftarrow p} |m, s; \vec{p}, \lambda\rangle = \Lambda_{p' \leftarrow p} \Lambda_p |m, s; \vec{0}, s_3 = \lambda\rangle$$

$$= \Lambda_{p'} \underbrace{(\Lambda_{p'}^{-1} \Lambda_{p' \leftarrow p} \Lambda_p)}_{} |m, s; \vec{0}, s_3 = \lambda\rangle$$

$\bar{p} \to p \to p' \to \bar{p}$ Wigner rotation R_W
that leaves rest frame invariant: $SO(3)$
(stability or little group) ($\hat{=} SU(2)/\mathbb{Z}_2$)

rotations on rest frame = $D^{(s)}$ irreps

$$= \Lambda_{p'} \sum_{\lambda'} D^{(s)}_{\lambda' \lambda}(R_W) |m, s; \vec{0}, s_3 = \lambda'\rangle$$

$$= \sum_{\lambda'} D^{(s)}_{\lambda' \lambda}(R_W) |m, s; \vec{p}', s_3 = \lambda'\rangle$$

"induced representation": Lorentz trafo $\Lambda_{p'}$ obtained from little group (= sub group)

Massless representations $m = 0$

- $p^2 = p^\mu p_\mu = 0 \Rightarrow \nexists$ rest frame $\Rightarrow$ quantum numbers and invariance group different!

choose standard frame
$$w_\mu = \hbar w(1, \underbrace{0, 0}_{\vec{\perp}}, 1)$$

invariance = stability group = {rotations & translations in 2 dim}
$\cong \underline{E_2 \text{ Euclidean group in 2 dim}}$

generators X_3 (clear: rot. along $3 = z$-axis)

and $L_1 = X_1 + Y_2$, $L_2 = X_2 - Y_1$ (combi of boost & rotations!!)

with algebra
$$\boxed{\begin{aligned} [X_3, L_1] &= iX_2 - iY_1 = iL_2 \\ [X_3, L_2] &= -iX_1 - iY_2 = -iL_1 \\ [L_1, L_2] &= iX_3 - iX_3 = 0 \end{aligned}}$$

- one can show that $[X_3, P_\mu]$, $[L_{1,2}, P_\mu]$ acting on w_μ give 0 (due to $P_{1,2}$ giving 0 and $P_3 - P_0$ giving 0, see below) and thus commute

$\Rightarrow$ states are still characterised by w_μ,
and helicity remains a good quantum number:

$\Rightarrow$ states are $|w_\mu, \lambda\rangle$
(= Lorentz invariant)

Note that λ cannot be changed here as in contrast to $m \neq 0$ we can no longer overtake a particle $\Rightarrow \lambda = \pm 1$ also called polarization.

- because of $\vec{x} \to -\vec{x}$, $\vec{p} \to -\vec{p}$, $\vec{J} = \vec{x} \times \vec{p} \to +\vec{J}$ under parity
$\vec{J} \cdot \vec{P} \to -\vec{J} \cdot \vec{P}$ under parity, so antiparticles have opposite λ

example: ν: $\lambda = -\frac{1}{2}$ massless neutrino
$\bar{\nu}$ $+\frac{1}{2}$ " anti ν

check using p. 127:

$$[L_1, P_0] = [(X_1 + Y_2), P_0] = iP_2 \quad \to 0 \text{ on } \omega_\mu$$

$$[L_1, P_j] = [X_1, P_j] + [Y_2, P_j] = -i\varepsilon_{j1k} P_k - i P_0 \delta_{j2}$$

$$= \begin{cases} j=1 & 0 \\ j=2 & i(P_3 - P_0) \\ j=3 & -iP_2 \end{cases}$$

so $[L_1 P_\mu] = 0$ on $|\omega_\mu, \lambda\rangle$

ditto for $[L_2, P_\mu] = 0$ —"—

Casimir operator for E_2: $\quad L_1^2 + L_2^2 \quad$ as $[X_3, L_1^2 + L_2^2] = \ldots = 0$

- for the algebra of E_2 we can define step operators $L_\pm = L_1 \pm i L_2$

$$\Rightarrow [X_3, L_\pm] = \pm L_\pm$$

* However, unlike for SU(2) where $\langle J_3^2\rangle \leq \langle \vec{J}^2\rangle$ we have no relation between X_3 and $L_1^2 + L_2^2$! $\Rightarrow$ we could generate an ∞ tower of states with $L_\pm$, so the only physical realisation is

$$L_1 |\omega_\mu, \lambda\rangle = 0 = L_2 |\omega_\mu, \lambda\rangle$$

$\Rightarrow$ Pauli-Lubanski:

$$W_\mu = -\frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} L^{\nu\rho} \omega^\sigma = \omega(X_3, L_1, L_2, X_3)$$

and thus $W^2 = W^\mu W_\mu = -\omega^2 (L_1^2 + L_2^2)$

we have $W_\mu |\omega_\mu, \lambda\rangle = \lambda\omega(1,0,0,1) |\omega_\mu, \lambda\rangle = \lambda P_\mu |\omega_\mu, \lambda\rangle$

and $W^2 |\omega_\mu, \lambda\rangle = 0$

here the induced rep is $\Lambda_{p' \leftarrow p} |p_\mu, \lambda\rangle = e^{-i\phi_W \lambda} |p'_\mu, \lambda\rangle$

angle wrt. J_3 in stability group.

* for $m=0$ states spin $\to$ helicity as we don't have an SU(2) rep!