Lecture Notes on "Integrable Systems"

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in E5-129

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Lecture Wednesdays 10^{15}-11^{45} D6-135

Fridays 12^{30}-14^{00} D6-135

Exercises Fridays ?

(from week 2 onwards)

Lecture notes will be posted on my webpage as .pdf files

Literature for Integrable Systems:

A. Das: Integrable models
World Scientific, Singapore 1989, 342 pages
FB 17 QD140 D229

Drazin, Philip G.: Solitons
Cambridge: Univ. Pr., 1983. - 136 S.
(London Mathematical Society lecture note series ; 85)
FB 10 QB433 D769

Ushveridze, Alexander G.: Quasi-exactly solvable models in quantum
mechanics / Alexander G. Ushveridze
Bristol [u.a.]: Inst. of Physics Publ., 1994. - XIV, 465 S. : graph. Darst.
FB 17 QD800 U85

L.D. Faddeev: Les Houches lecture notes, arXiv:hep-th/9605187v1

Integrable Systems

- organisation: - my coordinates
  - exam: if needed oral
  - Lecture times ok? Mo, Wed 8-10?
  - Exercise times: move Wed 16-18 → 14-16^00 (or other day?)
  - Literature

today
- intro general overview of content, why interesting

- the Korteweg-de Vries eq. (KdV)

this course: classical & quantum integrability

of discrete and continuous 2D (1 space & 1 time dim D)

non-linear PDE e.g. • $\boxed{\dot{u} = uu' + u'''}$ KdV

further examples • $\ddot{u} - u'' = e^{u}$ Liouville eq.

• $\ddot{u} - u'' + m^2 \sin u = 0$ Sine-Gordon (gen. Klein-Gordon + cos potential)

• $i\dot{\psi} + \psi'' - 2\kappa|\psi|^2\psi = 0$ Non-Lin. Schrödinger eq.

discrete examples

• $\dot{p}_i = e^{-(q_i - q_{i-1})} - e^{-(q_{i+1} - q_i)}$

$\dot{q}_i = p_i$ Toda-lattice eq. (discrete Liouville)

• XXX Heisenberg-Spin chain

Hamiltonian $\mathcal{H} = \sum_{\alpha, n} J^{\alpha} S_n^{\alpha} S_{n+1}^{\alpha}$

$S^{\alpha} = \sum_n S_n^{\alpha}$ total spin

quasi exactly solvable systems • anharmonic oscillator

Why is 2D special?

- the solvability of such non-linear PDE $\longleftrightarrow$ the existence of ∞ many conserved quantities

  this is only possible in general in 2D
- e.g. the dimension of the symmetry group of conformal transformations (translations, rotations, rescalings etc.) is ∞ only in 2D

Why is 2D interesting?

- 2D eqs. we study display soliton solutions
  = stable wave solutions that are localised and travel dispersionless in time
- $\exists$ physical situations where 2D is appropriate
  e.g. surfaces, films or wires (+t) in cond-matter applications
- idea string theory (for quantum gravity):
  replace world lines by 2D surfaces = world sheet
  $\rightarrow$ study (quantum) field theory on 2D $\rightarrow$ conformal field theory CFT
- systems with 2nd order phase transitions show long range correlation & universal critical exponents $\leftrightarrow$ CFT
- integrable spin chains appear in 4D $N=4$ supersymmetric Yang-Mills theory
- 2D theories can sometimes be solved non perturbatively

solution of non-lin PDE :

- $\exists$ variety of techniques (see list below)
- the KdV will serve as an example of reference to explain most of these :

• classical integrability :

Hamiltonian Mechanics $\to$ construct $\mathcal{H}$ - formulation of KdV
$\to$ find all conserved quantities of KdV

• inverse scattering method

- the solution $u(x,t)$ of the non lin PDE (e.g. KdV) is viewed as the potential of the (linear) time-independent Schrödinger eq.

$$\left(\partial_x^2 + u(x,t)\right)\psi(x) = -E\,\psi(x)$$

scattering wrt u :

$$\psi(x,t) \xrightarrow[x\to-\infty]{} e^{ikx} + \underset{\text{reflection}}{R(k,t)}\,e^{-ikx}$$

$$\xrightarrow[x\to+\infty]{} e^{ikx}\,\underset{\text{transmission}}{T(k,t)}$$

idea : - given $u(x,t=0)$ (a trivial sol.) $T(k,t=0)$, $R(k,t=0)$ follow

- if we can obtain the time evolution $T(k,t)$, $R(k,t)$ and reconstruct the corresp. $u(x,t)$ ("inverse scattering") we may find a non trivial solution $u(x,t)$ of KdV

- Lax Pair

given the non-lin PDE find a Lax Pair L, B describing a linear (matrix) eq

$$L(t)\,\Psi(t) = -\lambda\,\Psi(t)$$
$$\partial_t \Psi(t) = B(t)\,\Psi(t)$$

such that $\partial_t L = [B, L] \iff$ non-lin PDE

this commutator relation can be mapped to a

zero curvature condition $[\partial_t B, \partial_x - A] = 0$
(vanishing field strength tensor) $\iff \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = F_{\mu\nu} = 0$
the solution of which is $F_{\mu\nu}$ = pure gauge

- Bäcklund transformation:

- consists of a nontrivial, non-lin map from solution 1 $\downarrow$ solution 2
- it is equivalent to constructing action angle var.
- may permit to construct nontrivial from trivial solutions, or multi-solitons from single soliton ones

- Bethe Ansatz Equation (BAE)

example anharmonic oscillator $\mathcal{H} = -\partial_x^2 + Ax^2 + Bx^4 + Cx^6$

ansatz $\Psi(x) = \prod_i^M (x - \xi_i)\, e^{-ax^4 - bx^2}$

$\rightsquigarrow$ set of algebraic eqs determining ξ_i for the first M energy levels (higher levels not known).

BAE also extensively used in quantum integrable spin chains

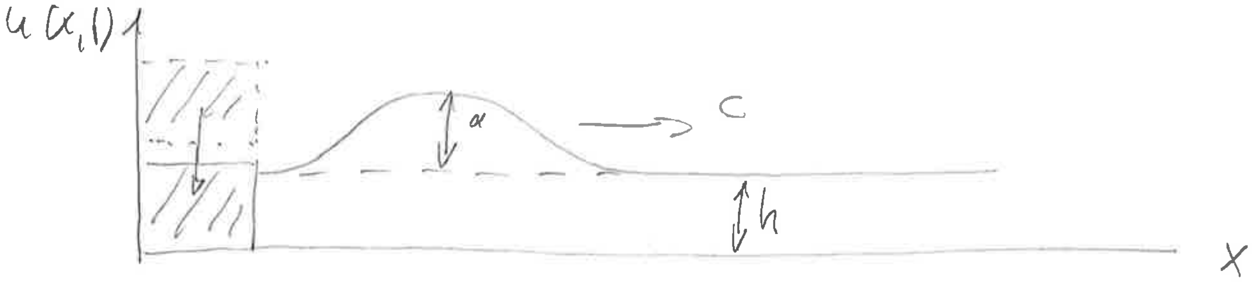
The Korteweg – deVries Equation [Phil. Mag 39 (1895) 422]

- study initiated by J.S. Russell observing a solitonic water wave along the canal Edinburgh-Glasgow 1834.

  quote

- Russell made experiments and established empirically

$$c = g(h+a)$$



Boussinesq & Rayleigh : $u(x,t) = \dfrac{a}{\cosh\left(\frac{x-ct}{b}\right)^2}$, $b^2 = 4h^2 \dfrac{(h+a)}{3a}$

based on the theory of incompressible fluids (assumption: length $\gg h$

KdV: diff. eq. with $u(x,t)$ as solution:

$$\partial_t u = \frac{3}{2}\sqrt{\frac{g}{h}}\left(u\partial_x u + \frac{2}{3}\alpha\partial_x u + \frac{1}{3}\sigma\,\partial_x^3 u\right)$$

where α const, $\sigma = \sigma(T, \rho)$ T surface tension, ρ density

more details : P.G. Drazin, original paper by KdV

equivalent forms of KdV

change coord $x \to x + vt$ $\Rightarrow$ $\partial_t u \to \partial_t u + v\partial_x u$

$\Rightarrow$ absorb linear term with $v = \alpha\sqrt{\frac{g}{h}}$

$$\Rightarrow \quad \boxed{\partial_t u = A u\partial_x u + B\partial_x^3 u}$$

Exercises: – directly verify by explicit differentiating that $u(x,t)$ given above solves this eq. at which values of A and B?

Symmetries of KdV

Exercise: verify Galilei invariance, translational inv. and scale invariance $x \to c^{\delta}$, $t \to c^{\gamma} t$, $u \to c^{\mu} u$

- we may choose $A = B = 1$ in the following using the rescaling $x \to B^{\frac{1}{3}} x$: $\partial_t u = A B^{-\frac{1}{3}} u \partial_x u + B \cdot B^{-1} \partial_x^3 u$

$u \to B^{\frac{1}{3}} A^{-1} u$: $\frac{B^{\frac{1}{3}}}{A} \partial_t u = \frac{A}{B^{\frac{1}{3}}} \cdot \frac{B^{\frac{2}{3}}}{A^2} + \frac{B^{\frac{1}{3}}}{A} \partial_x^3 u$

to get $$\boxed{\partial_t u = u \partial_x + \partial_x^3 u}$$

note: we will study other transformations later (modified KdV etc)

Questions:

1) are there other solutions, and if so how many?

2) given 2 solutions f_1, f_2 is $\alpha f_1 + \beta f_2$ a solution (superposition)?

3) uniqueness: given $u(x, t = t_0)$, is the solution uniquely determined $\forall t > t_0$?

Compare to $a_k f^{(k)}(x) + a_{k-1} f^{(k-1)}(x) + \dots a_0 f(x) = 0$ linear ODE with const. coeff.

to 1): $\exists$ k lin. indep solutions

to 2): Yes

to 3): solution uniquely determined through k indep. b.c.

- Questions 1) & 2) are difficult, we'll get back later

to 3) <u>proof of uniqueness</u>, by contradiction:

Suppose $\exists$ 2 solutions $u(x,t)$, $v(x,t)$ with the following properties

- $u(x,t)$, $v(x,t) \to 0$ for $x \to \pm\infty$ $\quad$ ($\Rightarrow w \to 0$ for $x \to \pm\infty$)
- $u(x,0) = v(x,0)$
- $\left\{ \begin{matrix} |u(x,t)|, & |v(x,t)| \\ |\partial_x u|, & |\partial_x v| \end{matrix} \right\} < \infty$ $\forall x$ and $\forall t$

we will show that then $w(x,t) := u(x,t) - v(x,t) \equiv 0$ $\forall t \geq 0$

Diff eq. for w:

it holds $\quad \partial_t u - \partial_t v = u\partial_x u - v\partial_x v + \partial_x^3 u - \partial_x^3 v$

add $\pm u\partial_x v$: $\quad \underline{\partial_t w = u\partial_x w + w\partial_x v + \partial_x^3 w}$

- w, integrate over $\int dx$:

lhs $\quad \int dx\, w\partial_t w = \frac{d}{dt}\int dx\, \frac{1}{2} w^2$

rhs $\quad \int dx \left[\frac{u}{2}\partial_x w^2 + w^2 \partial_x v + w\partial_x^3 w \right]$

integration by parts gives 0 for this term:

$$\int dx\, w\partial_x^3 w = -\int dx\, \partial_x w\, \partial_x^2 w = -\frac{1}{2}\int dx\, \partial_x (\partial_x w)^2 = 0$$

$$\Rightarrow \frac{d}{dt}\int dx\, \frac{1}{2} w^2 \overset{\text{int. by parts}}{=} \int dx\, w^2 \left[-\frac{1}{2}\partial_x u + \partial_x v \right]$$

define $E(t) \equiv \int dx\, \frac{1}{2} w(x,t)^2 \geq 0$; $m = 2 \max\limits_{x,t} \left[\partial_x v - \frac{1}{2}\partial_x u \right]$

$$\Rightarrow \frac{d}{dt} E(t) \leq m\, E(t) \quad \to 0 \leq E(t) \leq E(0)\, e^{mt}$$

but: for $t=0$ we have $E(t=0) = \int dx \frac{1}{2} w(x)^2 = \int dx \frac{1}{2}(u(x,0)-v(x,0))^2 = 0$

$\Rightarrow$ $\underline{E(t) \equiv 0}$ $\forall t \geq 0$ $\square$

Direct integration of the KdV

here choose coeff. A & B s.th.

$$\boxed{\dot{u} - 6uu' + u''' = 0}\ , \quad \cdot \hat{=} \partial_t, \ ' \hat{=} \partial_x$$

ansatz $f(z = x + ct) = u(x,t)$:

$$\Rightarrow \quad -cf' - 6ff' + f''' = 0 \iff f''' - 3(f^2)' - cf' = 0$$

$$\int dx \Rightarrow \quad f''(z) = 3f^2(z) + cf(z) + \alpha$$

we have $\frac{1}{2}\left[(f')^2\right]' = f'' \cdot f' \iff \frac{1}{2}\frac{d}{dx}\left[(f')^2\right] = f''\frac{df}{dx}$

$$\iff f'' = \frac{d\,\frac{1}{2}(f')^2}{df}$$

separation of variables for

$$d\,\tfrac{1}{2}(f')^2 = df\,(3f^2 + cf + \alpha)$$

$$\overset{\int}{\Rightarrow} \quad \tfrac{1}{2}(f')^2 = f^3(z) + \frac{c}{2}f^2(z) + \alpha f(z) + \beta \equiv F[f]$$

$$f' = \frac{df}{dz} \iff z = \int dz = \int \frac{df}{f'}$$

$\Rightarrow$ implicit solution $\quad z = \int \frac{df}{\pm 2\sqrt{F[f]}}$

i) boundary conditions for solitons: $f(z), f'(z), f''(z) \to 0$ for $z \to \pm\infty$

$$\Rightarrow \alpha = \beta = 0$$

$$\Rightarrow \quad z = \pm \int \frac{df}{f\sqrt{2f+c}}$$

solution is given by $\quad f(z) = \dfrac{-\frac{c}{2}}{ch^2\left[\frac{\sqrt{c}}{2}(z-z_0)\right]}$

ii) general solution:

$$F[f] = (f-f_1)(f-f_2)(f-f_3)$$ polynomial of degree 3 in f

with distinct zeros $f_3 < f_2 < f_1$:

$$z = z_3 \pm \int_{f_3}^{f} df \, \frac{1}{\sqrt{2F[f]}}$$ is an elliptic integral of the 1st kind (→ Abramowitz-Stegun)

with general solution

$$\underline{f(z) = f_2(z) - (f_2 - f_3)\, cn^2\left[\sqrt{\tfrac{1}{2}(f_1-f_3)}\,(z-z_3);\, m\right]}$$

Jacobi-elliptic function with modulus $m = \dfrac{f_2-f_3}{f_1-f_3}$

Hamiltonian Mechanics and its Application to KdV

- Hamiltonian mechanics describes the time evolution via the

  Lagrange function $\mathcal{L}(q,\dot{q},t)$ together with the

  Euler-Lagrange eq. of motion $\frac{\partial \mathcal{L}}{\partial q^i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i} = 0 \quad i=1,\dots,n$

  with coordinates $q^1 \dots q^n$, $n = \#$ d.o.f.

  These follow when minimising the

  action $W[q,\dot{q}] = \int_{t_1}^{t_2} dt\, \mathcal{L}(q,\dot{q},t)$

- Legendre trafo

  Introducing canonically conjugated momenta $\boxed{p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}^i}}$ $p_i q$ indep

  we can introduce a Legendre trafo to define the

  Hamilton function $\mathcal{H}(q,p,t) = p_i \dot{q}^i - \mathcal{L}(q,\dot{q}(q,p),t)$

  (Einstein summation convention)

  here we assume that the change of variables

  $(q_i,\dot{q}_i) \to (q_i,p_i)$ is invertible $\dot{q}^i = \dot{q}^i(q,p)$

[Note: this is not always the case, e.g. for $\mathcal{L} = \frac{1}{2} q\dot{q}^2$,

or for the Dirac eq, or for theories with gauge symmetries: in general we may have constraints

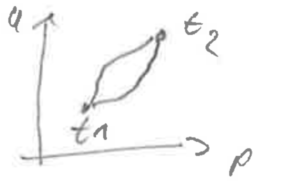
$$\phi_a(q,p,t) = 0$$

s.th. the phase space (q,p) is lower dimensional than $(q,\dot{q})$ (2n dim)

Hamiltonian eq. of motion

- these follow either by varying the Legendre transformed action

$$W[q,p] = \int_{t_1}^{t_2} dt \left(p_j \dot{q}^j - \mathcal{H}(q,p,t) \right)$$



s.th. the variations $\delta p, \delta q$ vanish at t_1, t_2:

$$\delta W = W[q+\delta q, p+\delta p] - W[q,p] \quad \text{, expand to 1. order:}$$

$$= \int_{t_1}^{t_2} dt \left(\delta p_i \dot{q}^i + p_i \delta \dot{q}^i - \delta q^i \frac{\partial \mathcal{H}}{\partial q^i} - \delta p_i \frac{\partial \mathcal{H}}{\partial p_i} \right)$$

$$= \int_{t_1}^{t_2} dt \left(\delta p_i \left(\dot{q}^i - \frac{\partial \mathcal{H}}{\partial p_i}\right) - \delta q^i \left(\dot{p}_i + \frac{\partial \mathcal{H}}{\partial q^i}\right) + \frac{d}{dt}\left(p_i \delta q^i\right) \right)$$

$$\Rightarrow \quad \boxed{\dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i}}$$

alternatively one can simply differentiate $\mathcal{H}$ & use Euler-Lagrange:

$$\frac{\partial \mathcal{H}}{\partial p_i} = \dot{q}^i + p_j \frac{\partial \dot{q}^j}{\partial p_i} - \frac{\partial \dot{q}^j}{\partial p_i} \underbrace{\frac{\partial L}{\partial \dot{q}^j}}_{\hat{=} p_j} = \dot{q}^i$$

$$\frac{\partial \mathcal{H}}{\partial q^i} = p_j \frac{\partial \dot{q}^j}{\partial q^i} - \frac{\partial L}{\partial q^i} - \frac{\partial \dot{q}^j}{\partial q^i} \underbrace{\frac{\partial L}{\partial \dot{q}^j}}_{\hat{=} p_j} = -\frac{\partial L}{\partial q^i} = -\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = -\dot{p}_i$$

- Poisson-bracket

for an arbitrary funct. of phase space variable (q,p) we have

$$\frac{d}{dt} A(q(t), p(t), t) = \frac{\partial A}{\partial q^i} \dot{q}^i + \frac{\partial A}{\partial p_i} \dot{p}_i + \frac{\partial A}{\partial t} \overset{\mathcal{H}\text{-eq.}}{=} \frac{\partial A}{\partial q^i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q^i} + \frac{\partial A}{\partial t}$$

$$\Rightarrow \quad \frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t}$$

we define the Poisson bracket w.r.t. canonical variables q, p as

$$\boxed{\{A, B\} \equiv \frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{q^i}}$$

Hence we can write $\quad \underline{\dfrac{dA}{dt} = \{A, \mathcal{H}\} + \dfrac{\partial A}{\partial t}}$

and the Hamilton eom. as

$$\boxed{\begin{aligned} \dot{q}^i &= \{q_i, \mathcal{H}\} = \frac{\partial \mathcal{H}}{\partial p_i} \\ \dot{p}_i &= \{p_i \mathcal{H} = -\frac{\partial \mathcal{H}}{\partial q^i} \end{aligned}}$$

using the indep of q_i and p_j. The

fundamental Poisson bracket read:
$$\boxed{\begin{aligned} \{q_i, q_j\} = 0 &= \{p_i, p_j\} \\ \{q_i, p_j\} &= \delta_{ij} \end{aligned}}$$

The Poisson bracket has the following properties:

- <u>anti symmetry</u> $\quad \{A, B\} = -\{B, A\}$
- <u>linearity</u> $\quad \{A+B, C\} = \{A, C\} + \{B, C\}$

  $\{cA, B\} = c\{A, B\}$ for all constants

hence $\quad \partial\{A, B\} = \{\partial A, B\} + \{A, \partial B\}$

for all <u>differentiations</u> $\partial = \partial_t, \partial_{q_i}, \partial_{p_i}$

- <u>product rule</u> $\quad \{A, BC\} = \{A, B\}C + B\{A, C\}$

- Jacobi identity $\quad 0 = \{A, \{B C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\}$

(proof later)

Example: n harmonic oscillators:

$$\mathcal{H}(q,p) = \sum_{i=1}^{n}\left(\frac{1}{2m}p_i^2 + \frac{1}{2}\omega^2 q_i^2\right)$$

$\mathcal{H}$ - e.o.m.:
$$\dot{q}_i = \{q_i, \mathcal{H}\} = \frac{\partial \mathcal{H}}{\partial p_i} = \frac{p_i}{m}$$
$$\dot{p}_i = \{p_i, \mathcal{H}\} = -\frac{\partial \mathcal{H}}{\partial q_i} = -\omega^2 q_i$$

Formulation as a symplectic manifold - discrete systems

defining $y^\mu = (q,p)$ $\quad \mu = 1,\dots,n$

with $y^i = q^i$ for $i=1,\dots,n$, $y^{i+n} = p_i$ $i=1\dots n$

we can write the following canonical Poisson brackets

$$\{y^\mu, y^\nu\} = \epsilon^{\mu\nu}, \qquad \epsilon^{\mu\nu} = \begin{pmatrix} 0 & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & 0 \end{pmatrix}$$

- defining $\partial_\mu = \frac{\partial}{\partial y^\mu}$ we have

$$\{A(y), B(y)\} = \epsilon^{\mu\nu}\partial_\mu A\, \partial_\nu B = \partial_\mu A \{y^\mu, y^\nu\}\partial_\nu B$$

and for the Hamilton e.o.m.

$$\boxed{\dot{y}^\mu = \{y^\mu, \mathcal{H}(y)\} = \epsilon^{\mu\nu}\partial_\nu \mathcal{H}}$$

Exercise: it holds $\{y^\mu, \{y^\nu, y^\lambda\}\} + \text{cyclic perm} = 0$

Jacobi identity

Example Harmonic Oscillator ($m \equiv 1$)

$$\mathcal{H}(y) = \sum_{i=1}^{n}\left(\frac{1}{2}(y^{i+n})^2 + \frac{1}{2}\omega^2 (y^i)^2\right) = \frac{1}{2}\omega \sum_{i=1}^{n}\left(\frac{(y^{i+n})^2}{\omega} + \omega (y^i)^2\right)$$

define $z^\mu = \left(\sqrt{\omega}\, y^i, \frac{1}{\sqrt{\omega}} y^{i+n}\right)$

$\Rightarrow \{z^\mu, z^\nu\} = \epsilon^{\mu\nu}$ (as $\{\sqrt{\omega}\, y^i, \frac{1}{\sqrt{\omega}} y^{j+n}\} = \delta^{ij}$

$\{\frac{1}{\sqrt{\omega}} y^{i+n}, \sqrt{\omega}\, y^j\} = -\delta^{ij}$)

so we obtain

$$\mathcal{H}(z) = \frac{1}{2}\omega \sum_{i=1}^{2n} (z^i)^2 ,$$

$$\dot{z}^\mu = \{z^\mu, \mathcal{H}\} = \epsilon^{\mu\nu}\partial_\nu \mathcal{H}(z)$$

$$= \epsilon^{\mu\nu}\partial_\nu\left(\frac{\omega}{2} z_\lambda z^\lambda\right) = \epsilon^{\mu\nu}\omega z_\nu$$

- in this formulation we have a duality $\sqrt{\omega} \leftrightarrow \frac{1}{\sqrt{\omega}}$

$y \leftrightarrow p$

Non-canonical Poisson brackets

suppose we have $\{y^\mu, y^\nu\} = f^{\mu\nu}(y)$, $f^{\mu\nu}(y) = -f^{\nu\mu}(y)$

anti-symmetric

and invertible $f_{\mu\lambda} f^{\lambda\nu} = \delta_\mu^\nu$ where $f_{\mu\lambda} = (f^{\mu\lambda})^{-1}$

we then have

$$\{A(y), B(y)\} = \partial_\mu A(y)\, f^{\mu\nu}(y)\, \partial_\nu B(y)$$

and $$\dot{y}^\mu = \{y^\mu, \mathcal{H}\} = f^{\mu\nu}(y)\, \partial_\nu \mathcal{H}(y)$$

the Jacobi identity $\{y^\mu,\{y^\nu,y^\lambda\}\}+$ cyclic $=0$

which continues to hold impose the following restriction on allow $f_{\mu\nu}=f^{-1}$: $\boxed{0=\partial_\mu f_{\nu\lambda}(y)+\partial_\nu f_{\lambda\mu}(y)+\partial_\lambda f_{\mu\nu}(y)=0}$

Bianchi identity

Exercise: show this and check $\{A,\{B,C\}\}+$ cyclic $=0$ in this generalized setting.

back to standard phase space $\{q,p\}$:

Conserved Quantities and Noether Theorem

a conserved quantity is a function of phase space $\phi(q,p,t)$ that doesn't change its value when $q(t),p(t)$ follow the physical trajectory in phase space (satisfy e.o.m.):

$$0=\frac{d}{dt}\phi=\{\phi,\mathcal{H}\}+\frac{\partial}{\partial t}\phi$$

When ϕ_1 and ϕ_2 are conserved quantities also linear combinations are (linearity of Poisson bracket), and due to the product rule any polynomial in ϕ_1 and ϕ_2 is conserved. However, this will not yield new conserved quantities. In contrast, the Poisson bracket may yield a new cons. quant.: $\frac{d}{dt}(\{\phi_1,\phi_2\})=\{(\{\phi_1,\phi_2\}),\mathcal{H}\}+\frac{\partial}{\partial t}(\{\phi_1,\phi_2\})$

$$\overset{\text{Jacobi}}{\underset{\text{diff}}{=}}\{\{\phi_1,\mathcal{H}\},\phi_2\}+\{\phi_1,\{\phi_2,\mathcal{H}\}\}+\{\frac{\partial\phi_1}{\partial t},\phi_2\}+\{\phi_1,\frac{\partial\phi_2}{\partial t}\}$$

$$=0$$

In general $\{\Phi_1, \Phi_2\}$ is not a polynomial funct. $f(\Phi_1, \Phi_2)$ of Φ_1 and Φ_2 (of course $\{\Phi_1, \Phi_2\} = 0$ gives nothing new!)

example: angular momentum

$$L_1 = y p_z - z p_y, \quad L_2 = z p_x - x p_z, \quad L_3 = x p_y - y p_x$$

satisfy $\{L_1, L_2\} = L_3$

Noether Theorem

"every continuous symmetry of the action corresponds to a conserved quantity"

- a symmetry is a trafo T and function $\hat{F}$

$$T: (q, p) \to (q'(q,p), p'(q,p))$$

that leaves the integrand of the action invariant (up to boundary terms)

$$\boxed{p_i \dot{q}^i - \mathcal{H}(q, p, t) = p_i' \dot{q}'^i - \mathcal{H}(q', p', t) + \frac{d\hat{F}}{dt}}$$

- T is continuous if it is a differentiable 1-parameter family T_α with $T_{\alpha+\beta} = T_\alpha \circ T_\beta$, $T_0 = Id$, $\exists$ inverse $T_{-\alpha}$ to T_α

- an infinitesimal trafo $(\delta q, \delta p)$ is given by $\partial_\alpha T_\alpha|_{\alpha=0}$

$$(\delta q, \delta p) = \partial_\alpha T_\alpha(q, p)|_{\alpha=0} = \partial_\alpha (q', p')|_{\alpha=0}$$

- denote $\partial_\alpha \hat{F}|_{\alpha=0}$ by F and differentiate the rhs

$$\Rightarrow 0 = \delta p_j \dot{q}^j + p_j \delta \dot{q}^j - \delta q^j \frac{\partial \mathcal{H}}{\partial q^j} - \delta p_j \frac{\partial \mathcal{H}}{\partial p_j} + \frac{d}{dt} F$$

$$= \delta p_j \dot{q}^j - \dot{p}_j \delta q^j - \delta q^j \frac{\partial \mathcal{H}}{\partial q^j} - \delta p_j \frac{\partial \mathcal{H}}{\partial p_j} + \frac{d}{dt} \underbrace{(F + p_j \delta q^j)}_{\equiv \phi}$$

$$= \dot{q}^j \left(\delta p_j + \frac{\partial \phi}{\partial p_j}\right) - \dot{p}_j \left(\delta q^j - \frac{\partial \phi}{\partial p_j}\right) - \delta q^j \frac{\partial \mathcal{H}}{\partial q^j} - \delta p_j \frac{\partial \mathcal{H}}{\partial p_j} + \frac{\partial \phi}{\partial t}$$

this has to vanish identically in all var $(q, p, \dot{q}, \dot{p})$

coeffs in $\dot{q}, \dot{p}$ $\Rightarrow$ $\boxed{\delta q^j = \frac{\partial \phi}{\partial p_j}, \quad \delta p_j = -\frac{\partial \phi}{\partial q^j}}$ ϕ generates infinitesimal trafos

$$\Rightarrow \quad 0 = -\frac{\partial \phi}{\partial p_j} \frac{\partial \mathcal{H}}{\partial q^j} + \frac{\partial \phi}{\partial q^j} \frac{\partial \mathcal{H}}{\partial p_j} + \frac{\partial \phi}{\partial t}$$

$$= \{\phi, \mathcal{H}\} + \frac{\partial \phi}{\partial t} = \frac{d\phi}{dt}$$ is conserved

- When writing the KdV equation as a Hamiltonian system we will construct an infinite set of indep. conserved quantities. This is in one to one correspondance to find an infinite set of symmetry transformations!

- <u>Action angle variables</u>: suppose we find new coordinates $\underline{P}_i = \underline{P}_i(q, p)$, $\theta_i = \theta_i(q, p)$ s.th. $\{\theta_i, \theta_j\} = 0 = \{\underline{P}_i, \underline{P}_j\}$, $\{\theta_i, \underline{P}_j\} = \delta_{ij}$ <u>and</u> $\mathcal{H} = \mathcal{H}(\underline{P})$ is θ-indep. Then

$\dot{\underline{P}}_i = \{\underline{P}_i, \mathcal{H}(\underline{P})\} = 0$ are conserved

$\dot{\theta}_j = \{\theta_j, \mathcal{H}(\underline{P})\} = -\frac{\partial \mathcal{H}}{\partial \underline{P}_j} = \text{const (in time)} \Rightarrow$ integrate

- the conserved quantities are independent:

$$\{ P_i , P_j \} = 0 \quad \text{also called in involution}$$

- here we have used that the Poisson bracket are coordinate independent (check as an exercise)

Poisson bracket for fields

take limit on an infinite number of 2n coordinates

$$y^{\mu} \xrightarrow[2n \to \infty]{} u(z) \,, \quad f^{\mu\nu}(y) \xrightarrow[2n\to\infty]{} f(x, z, u(x), u(z))$$

Poisson bracket $\{ u(x), u(z) \} = f(x,z) = - f(z,x)$

antisym.

using the functional derivative

$$\frac{\delta}{\delta u(x)} \quad \text{with} \quad \frac{\delta u(z)}{\delta u(x)} = \delta(x-z)$$

$$\frac{\delta}{\delta u(x)} \int_{-\infty}^{\infty} dz\, u^2(z) = \int_{-\infty}^{\infty} dz\, 2u(z)\, \delta(x-z) = 2u(x)$$

we define for functionals

$$\{ A[u], B[u] \} = \int_{-\infty}^{\infty} dx\, dz\, \frac{\delta A}{\delta u(x)} f(x,z) \frac{\delta B}{\delta u(z)}$$

with the Hamilton eq. o. m.

$$\dot{u}(x) = \{ u(x), \mathcal{H}[u] \} = \int_{-\infty}^{\infty} dz\, f(x,z) \frac{\delta \mathcal{H}}{\delta u(z)}$$

and Bianchi identity for f^{-1}

$$\partial_x f^{-1}(y,z) + \text{cyclic} = 0$$

Example: continuous chain of harmonic oscillators:

we had $\mathcal{H}(z) = \frac{1}{2}\omega \sum_{i=1}^{2n} (z^i)^2$

so $z^\mu \to u(x)$, $\mathcal{H}(z) \to \int_{-\infty}^{\infty} dx \, \frac{1}{2}\omega \, u(x)^2 \equiv \mathcal{H}[u]$

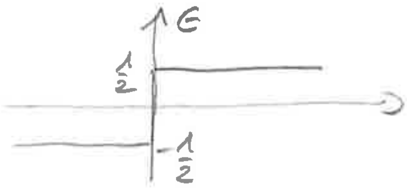
Q: what is the Poisson bracket in terms of f = ?

$$\dot{u}(x) = \{u(x), \mathcal{H}[u]\} = \int_{-\infty}^{\infty} dy \, f(x,y) \underbrace{\frac{\delta \mathcal{H}}{\delta u(y)}}_{\omega u(y)}$$

choose

$$f(x,y) = \epsilon(x-y) = \overset{\text{Heaviside}}{\Theta}(x-y) - \frac{1}{2}$$

canonical Poisson bracket



stepfunction

$$\Rightarrow \quad \dot{u}(x) = \int_{-\infty}^{\infty} dy \, \epsilon(x-y) \, \omega \, u(y)$$

e.o.m

apply ∂_x $\Rightarrow$ $\partial_x \partial_t u(x) = \omega \, u(x)$

inverse f: $f^{-1}(x,y) = \partial_x \delta(x-y)$ normalised to δ:

because $\int_{-\infty}^{\infty} dz \, f(x,z) \, f^{-1}(z,y) = \int_{-\infty}^{\infty} dz \, \epsilon(x-z) \, \partial_z \delta(z-y)$

$$\overset{\text{int by parts}}{=} - \int_{-\infty}^{\infty} dz \, \underbrace{\partial_z \epsilon(x-z)}_{-\delta(x-z)} \, \delta(z-y)$$

$$= \delta(x-y)$$

- here the Bianchi identity is trivially satisfied (check!)