Conformal symmetry in 2 dimensions

$d=2$: $\quad \partial^\alpha \epsilon^\beta(x) + \partial^\beta \epsilon^\alpha(x) = (\partial_\delta \epsilon^\delta(x))\, g^{\alpha\beta}$

consider Euclidean, flat space $\quad g_{\mu\nu} = \delta_{\mu\nu} = \delta^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\alpha = \beta = 1$: $\quad \partial^1\epsilon^1 + \partial^1\epsilon^1 = \partial^1\epsilon^1 + \partial^2\epsilon^2$, dillo for $\alpha=\beta=2$

$$\Leftrightarrow \quad \boxed{\partial_1 \epsilon_1(x) = \partial_2 \epsilon_2(x)} \quad (\Delta 1)$$

$\alpha = 1, \beta = 2$ $\quad \boxed{\partial_1 \epsilon_2(x) + \partial_2 \epsilon_1(x) = 0} \quad (\Delta 2)$ dillo for $\alpha = 2, \beta = 1$

choose complex coordinates

$$z = x^1 - i x^2 \qquad \text{and} \qquad \epsilon = \epsilon_1 - i\epsilon_2$$
$$\Rightarrow \bar{z} = x^1 + i x^2 \qquad\qquad \bar{\epsilon} = \epsilon_1 + i\epsilon_2$$

$$\Rightarrow \quad \underline{\partial_z} \, \epsilon_1(x_1, x_2) = \partial_z \, \epsilon_1\left(\tfrac{1}{2}(z+\bar{z}), \tfrac{1}{2i}(-z+\bar{z})\right)$$
$$= \tfrac{1}{2}\partial_1 \epsilon_1 - \tfrac{1}{2i}\partial_2 \epsilon_1 = \underline{\tfrac{1}{2}(\partial_1 + i\partial_2)}\,\epsilon_1$$

analogously $\quad \partial_{\bar{z}} = \tfrac{1}{2}(\partial_1 - i\partial_2)$

$\left(\text{and} \quad \partial_z z = 1 = \partial_{\bar{z}} \bar{z} \;, \quad \partial_z \bar{z} = 0 = \partial_{\bar{z}} z \right)$

$$\Longrightarrow \quad \boxed{\partial_z \bar{\epsilon}(z,\bar{z}) = 0} \quad \text{and} \quad \boxed{\partial_{\bar{z}} \epsilon(z,\bar{z}) = 0}$$

as $\quad \begin{Bmatrix} \partial_{\bar{z}} \epsilon \\ \partial_z \bar{\epsilon} \end{Bmatrix} = \tfrac{1}{2}(\partial_1 \mp i\partial_2)(\epsilon_1 \mp i\epsilon_2)$

$$= \tfrac{1}{2}\left(\partial_1\epsilon_1 - \partial_2\epsilon_2 \mp i(\partial_1\epsilon_2 + \partial_2\epsilon_1)\right) = 0$$

$\Rightarrow$ <u>any</u> function $\epsilon = \epsilon(z)$ is a solution!
$\bar{\epsilon} = \bar{\epsilon}(\bar{z})$

Infinitesimal generators

$$L_n = -z^{n+1}\partial_z \quad : \quad z \to z' = z - z^{n+1}$$
$$\bar{L}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \quad\quad \bar{z} \to \bar{z}' = \bar{z} - \bar{z}^{n+1}$$

$\left.\right\}$ $n \in \mathbb{Z}$ so ∞ many

$\Rightarrow$ the generator of an arbitrary, infinitesimal conformal trafo in z is given by the following Laurent series

$$\sum_{n\in\mathbb{Z}} \epsilon_n L_n = \sum_{n\in\mathbb{Z}} -\epsilon_n z^{n+1}\partial_z$$

• the generators L_n and $\bar{L}_n$ form an ∞-dim algebra:

$$[L_n, L_m] = (n-m) L_{m+n} \qquad n, m \in \mathbb{Z}$$
$$[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{m+n} \qquad \text{— " —}$$
$$[L_n, \bar{L}_m] = 0$$

(classical) Virasoro algebra

because: $z, \bar{z}$ are indep so $[\partial_z, \partial_{\bar{z}}] = 0$, $\partial_z \bar{z} = 0 = \partial_{\bar{z}} z$

and $L_n L_m = -z^{n+1}\partial_z (-) z^{m+1}\partial_z = (m+1) z^{n+m+1}\partial_z + z^{n+m+2}\partial_z^2$

$$\Rightarrow [L_n, L_m] = \big((m+1) - (n+1)\big) z^{n+m+1}\partial_z + 0$$
$$= -(m-n) L_{n+m}$$

and ditto for $\bar{L}_n$'s

- as we have seen $\{L_{-1}, L_0, L_{+1}\}$ form a finite dimensional subalgebra, it is isomorphic to $su(2)$
- other subalgebras are $\{L_n \mid n \geq l\}$ with $l = -1, 0, 1, \ldots$
  or $\{L_n \mid n \leq l\}$ with $l = +1, 0, -1, \ldots$

(ex: can you find other finite-dim subalgebras?)

- the generators $P_\mu, M_{\mu\nu}, D, K_\mu$ of the conformal group in $d > 2$ dimensions p. 96 can be expressed solely with

$$L_{-1} = -z^0 \partial_z = -\partial_z$$

$$L_0 = -z \partial_z$$

$$L_{+1} = -z^2 \partial_z$$

- <u>finite transformations</u>:

- the $su(2)$ subalgebra generate the following finite trafo by successive application

	infinit.	finite
L_{-1}	$z \to z + \epsilon_{-1} \to z + \epsilon_{-1} + \epsilon'_{-1} \to$	$z + \alpha$
L_0	$z \to z + \epsilon_0 z \to (1+\epsilon_0)(1+\epsilon_0') z + \ldots$	λz
L_{+1}	$z \to z + \epsilon_1 z^2 \to z + \epsilon_1' z^2 + \epsilon_1 (z + \epsilon_1' z^2)^2 \to \ldots$ $= z(1 + \epsilon_1 z)$	$\to \dfrac{z}{1 + \beta z}$

- the combination of these gives

$$z \to \frac{az + b}{cz + d} \quad \text{with } ad - bc = 1 \qquad \text{Möbius trafos}$$

- these form the group of (rational) linear trafos

Bäcklund Transformations (BT)

- BT can be used to generate:
  - nontrivial solutions from trivial ones
  - multi soliton solutions from single soliton solutions

- a BT is defined by a change of variables $u(x,t) \to v(x,t)$ given by the set of 1st order eqs in u:

$$\partial_x u(x,t) = F(u(x,t), v(x,t))$$

$$\partial_t u(x,t) = G(u(x,t), v(x,t))$$

such that the eq. for $v(x,t)$ is

i) identical to the initial nonlinear PDE (for $u(x,t)$)

ii) simpler than the initial eq. and can be solved

Applications:

to i) example Sine-Gordon

$$(\partial_t^2 - \partial_x^2)\, u(x,t) = \sin(u(x,t)) \xleftrightarrow{BT} (\partial_t^2 - \partial_x^2)\, v(x,t) = \sin(v(x,t))$$

$\to$ we will show how to generat a nontrivial solution $u(x,t) \neq 0$ from a trivial one, $v(x,t) \equiv 0$, by inserting this $v \equiv 0$ into BT

to i) we will show for KdV & Sine-Gordon how to generate a 2-soliton solution in $u(x,t)$ trought a 1 soliton solution in $v(x,t)$ by inserting into BT to ii) example Liouville eq.

$$(\partial_t^2 - \partial_x^2)\, u(x,t) = e^{u(x,t)} \quad \xleftrightarrow{BT} \quad (\partial_t^2 - \partial_x^2)\, v(x,t)$$

free wave eq.

note: there exists a systematic way to construct BT going back to J. Clairin – it is not always straightforward

The Liouville Equation (J. Liouville 1853)

modern application in string theory:

- the string action has a 2-dim scale invariance $z \to \lambda z$
  $\Rightarrow$ when quantising the theory the scale parameter becomes a field $\lambda = e^{u(x,t)}$ that satisfies the Liouville eq.
- here we will consider classical solutions of -"- only!

$x = x^+,\ t = -x^-$

in light cone coordinates $x^{\pm} = \frac{1}{2}(\tau \pm \gamma)$ see p71

the Liouville eq. becomes $\boxed{\partial_+ \partial_- u(x^+, x^-) = \exp[u(x^+, x^-)]}$

[compared to Sine-Gordon $\partial_+ \partial_- u = \sinh u = \frac{1}{2}(e^{u} - e^{-u})$]

Bäcklund-trafo $u(x^+, x^-) \to v(x^+, x^-)$ for Liouville

$$\boxed{\begin{aligned} &1. \quad \partial_+ u = -\partial_+ v + \alpha e^{\frac{1}{2}(u-v)} \\ &2. \quad \partial_- u = \partial v + \frac{2}{\alpha} e^{\frac{1}{2}(u+v)} \end{aligned}}$$

where $\alpha = \text{const} \neq 0$ is a parameter $\in \mathbb{R}$

this leads to the free wave eq. for $v(x^+,x^-)$:

$$\bullet\ \partial_-\ 1. \Rightarrow \quad \partial_-\partial_+ u = -\partial_-\partial_+ v + \frac{\alpha}{2} e^{\frac{1}{2}(u-v)} \underbrace{(\partial_- u - \partial_- v)}_{2.\ \frac{2}{\alpha} e^{\frac{1}{2}(u+v)}}$$

$$= -\partial_-\partial_+ v + e^{u}$$

so when u satisfies Liouville v satisfied

$$\boxed{0 = \partial_-\partial_+ v(x^+, x^-)}$$ free wave eq. (in light cone coord.)

• analogously from ∂_+ 2. and 1. we obtain $0 = \partial_+\partial_- v$ (ex.)

general solution for 2-dim free wave eq.:

$$v(x^+, x^-) = f(x^+) + g(x^-)$$

where f, g arbitrary smooth & differentiable functions

• Solution for v & BT $\Rightarrow$ solution $u(x,t)$:

$v = f + g$ into BT 1.:

$$\partial_+ u = -\partial_+ (f(x^+) + g(x^-)) + \alpha e^{\frac{1}{2}(u-f-g)}$$

$$\Leftrightarrow \partial_+ (u + f(x^+) - g(x^-)) = \alpha e^{\frac{1}{2}(u-f-g)} \qquad \Big| \cdot e^{-\frac{1}{2}(u+f-g)}$$

add 0

$$\Leftrightarrow (-2)\partial_+ \left(e^{-\frac{1}{2}(u+f-g)}\right) = \alpha e^{\frac{1}{2}(\cancel{u}-f-\cancel{g}-\cancel{u}-f+\cancel{g})} = \alpha e^{f(x^+)}$$

$\cdot\left(-\frac{1}{2}\right)$, integration wrt x^+ $\Rightarrow$

$$e^{-\frac{1}{2}(u(x^+,x^-)+f(x^+)-g(x^-))} = -\frac{\alpha}{2}\int_c^{x^+} dx' e^{f(x')} + a(x^-)$$

x^--dep int. const

* <u>analogously for $V = f + g$ in BT 2:</u>

$$\partial_- u = \partial_-(\cancel{f(x^+)} + g(x^-)) + \frac{2}{\alpha} e^{\frac{1}{2}(u+f+g)}$$

$$\Leftrightarrow \partial_-(u - g + \underset{\text{add } 0}{f}) = \frac{2}{\alpha} e^{\frac{1}{2}(u+f+g)} \qquad \Big| \cdot e^{-\frac{1}{2}(u+f-g)}$$

$$\Rightarrow (-2)\partial_- e^{-\frac{1}{2}(u+f-g)} = \frac{2}{\alpha} e^{g}$$

$$\Rightarrow \qquad \underline{e^{-\frac{1}{2}(u+f-g)} = -\frac{1}{\alpha} \int_c^{x^-} dx' e^{g(x')} + b(x^+)}$$

setting both rhs equal and using that x^+ and x^- are indep.

we obtain

$$\boxed{\begin{aligned} a(x^-) &= -\frac{1}{\alpha} \int_c^{x^-} dx' e^{+g(x')} \\ b(x^+) &= -\frac{\alpha}{2} \int_c^{x^+} dx' e^{-f(x')} \end{aligned}}$$

and thus $\quad \exp[-\tfrac{1}{2}(u+f-g)] = a + b$

$$\Leftrightarrow \quad \boxed{u(x^+, x^-) = \underbrace{-f(x^+) + g(x^-)}_{\text{sol of the homog. eq. } \partial_+\partial_- u = 0} - 2\ln[a(x^-) + b(x^+)]}$$

[check that this is a solution:

$$\partial_+\partial_- u = -2\partial_+\left[\frac{\partial_- a(x^-)}{a(x^-) + b(x^+)}\right] = 2\,\frac{\partial_+ b(x^+)\partial_- a(x^-)}{(a+b)^2}$$

$$= 2\,\frac{(-\frac{\alpha}{2})\, e^{-f(x^+)} \cdot (-\frac{1}{\alpha})\, e^{+g(x^-)}}{(a+b)^2} = \frac{e^{-f+g}}{(a+b)^2}$$

and $\quad e^u = e^{-f+g-2\ln[a+b]} = \dfrac{e^{-f+g}}{(a+b)^2}$

ok]