The Sine - Gordon Equation

- example of a Bäcklund trafo onto inself (type i.)

Sine-Gordon $\quad (\partial_\tau^2 - \partial_y^2)\, u(y,\tau) = \sin(u(y,\tau))$

or $\quad \boxed{\partial_+ \partial_- u(x^+,x^-) = \sin(u(x^+,x^-))}$ in light cone coordinates

Bäcklund - transformation :

$$\boxed{\begin{aligned} &1.) \quad \partial_+ u = \partial_+ v + 2\alpha \sin\left(\frac{u+v}{2}\right) \\ &2.) \quad \partial_- u = -\partial_- v - \frac{2}{\alpha} \sin\left(\frac{v-u}{2}\right) \end{aligned}}$$

$0 \neq \alpha \in \mathbb{R}$ const

- very similar to BT. for Liouville's eq: exp $\to$ sine (except factor 2)

eqs. for $v(x^+,x^-)$:

$$\partial_- 1.): \quad \partial_-\partial_+ u = \partial_-\partial_+ v + 2\alpha \cos\left(\frac{u+v}{2}\right) \frac{1}{2}(\partial_- u + \partial_- v)$$

$$\overset{2.)}{=} \partial_-\partial_+ v + 2\alpha \cos\left(\frac{u+v}{2}\right)\left(-\frac{1}{\alpha}\right) \sin\left(\frac{u-v}{2}\right)$$

$$= \partial_-\partial_+ v - 2\, \underbrace{\frac{1}{2}\left(e^{i\left(\frac{u+v}{2}\right)} + e^{-i\left(\frac{u+v}{2}\right)}\right) \frac{1}{2i}\left(e^{i\left(\frac{u-v}{2}\right)} - e^{-i\left(\frac{u-v}{2}\right)}\right)}_{= \frac{1}{2i}\left(e^{iu} - e^{iv} + e^{-iv} - e^{-iu}\right)}$$

$$\Leftrightarrow \quad \partial_-\partial_+ u = \partial_-\partial_+ v - \sin u + \sin v \qquad \sim = \sim \text{ identity}$$

u solution of sine-Gordon $\Rightarrow \partial_-\partial_+ v = \sin v$

- the same follows from $\partial_+ 2.)$ (check!)

$\Rightarrow$ this BT is a relation between 2 solutions u and V of Sine-Gordon

Examples (see below):

- choose the trivial solution for one, e.g. $u \equiv 0$

  $\Rightarrow$ non-trivial solution for the other: $V(x^+, x^-)$ (1 solition)

- choose a 1 soliton solution for V $\Rightarrow$ BT gives a 2 soliton solution for $u(x^+, x^-)$

1 Soliton Solution of Sine-Gordon:

$$u \equiv 0 \quad \Rightarrow \quad 0 = \partial_+ V + 2\alpha \sin\left(\frac{V}{2}\right)$$

$$0 = -\partial_- V - \frac{2}{\alpha} \sin\left(\frac{V}{2}\right)$$

define a rescaling $\tilde{x}^+ \equiv \alpha x^+ \Rightarrow \tilde{\partial}_+ = \frac{1}{\alpha}\partial_+$

$\tilde{x}^- \equiv \frac{1}{\alpha} x^- \Rightarrow \tilde{\partial}_- = \alpha \partial_-$

$$\Rightarrow \quad \boxed{\tilde{\partial}_+ V = -2\sin\left(\frac{V}{2}\right) = \tilde{\partial}_- V}$$

$\Rightarrow$ ansatz $V(\tilde{x}^+, \tilde{x}^-) = V(\tilde{x}^+ + \tilde{x}^-)$ as $(\tilde{\partial}_+ + \tilde{\partial}_-)V \neq 0$, $(\tilde{\partial}_+ - \tilde{\partial}_-)V = 0$

it holds $\tilde{\partial}_+ V = -2\sin\left(\frac{V}{2}\right) = -4\sin\left(\frac{V}{4}\right)\cos\left(\frac{V}{4}\right)$

$: \cos^2\left(\frac{V}{4}\right) \neq 0 \Big|$ for $\frac{V}{4} = \frac{\pi}{2} \bmod \pi$

$$\frac{1}{\cos^2\left(\frac{V}{4}\right)} \tilde{\partial}_+ V = -4\tan\left(\frac{V}{4}\right)$$

$$\Leftrightarrow \quad 4\tilde{\partial}_+\left(\tan\left(\frac{V}{4}\right)\right) = -4\tan\left(\frac{V}{4}\right)$$

exp-diff eq. for $\tan\frac{V}{4}$ in $\tilde{\partial}_+$, ditto in $\tilde{\partial}_-$

$$\Rightarrow \quad \tan\left(\frac{V(\tilde{x}^+ + \tilde{x}^-)}{4}\right) = c\,\exp\left[-(\tilde{x}^+ + \tilde{x}^-)\right]$$, c from boundary cond.

$$= c\,\exp\left[-\alpha \tfrac{1}{2}(\tau+y) - \tfrac{1}{\alpha}\tfrac{1}{2}(\tau - y)\right]$$

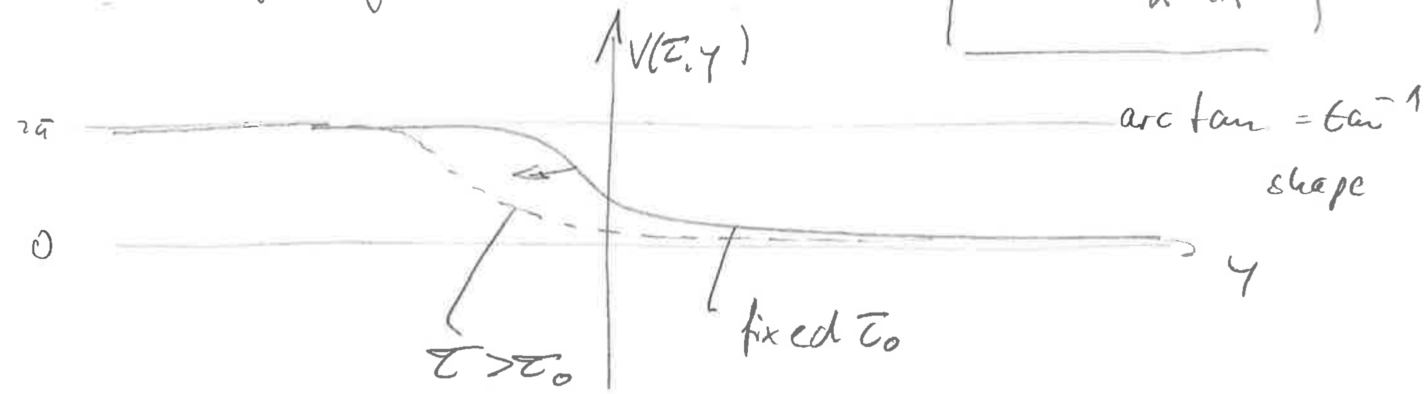
$$= c\,\exp\left[-\tfrac{1}{2}\left(\alpha+\tfrac{1}{\alpha}\right)\tau - \tfrac{1}{2}\left(\alpha - \tfrac{1}{\alpha}\right)y\right]$$

$$= c\,\exp\left[-\tfrac{1}{2}\left(\frac{\alpha^2+1}{\alpha}\right)\left(\tau + \left(\frac{\alpha^2-1}{\alpha^2+1}\right)y\right)\right]$$

$\Rightarrow$ solution

$$\boxed{V(\tau, y) = 4\tan^{-1}\left\{c\,\exp\left[-\tfrac{1}{2}\left(\frac{\alpha^2+1}{\alpha}\right)\left(\tau + \frac{\alpha^2-1}{\alpha^2+1}y\right)\right]\right\}}$$

- one-parameter family of solutions propagating with inverse velocity $\boxed{\vartheta^{-1} = \frac{\alpha^2-1}{\alpha^2+1}}$ "1 kink" solution $= 1$ soliton



- moving left for $c > 0$, $\vartheta > 0$

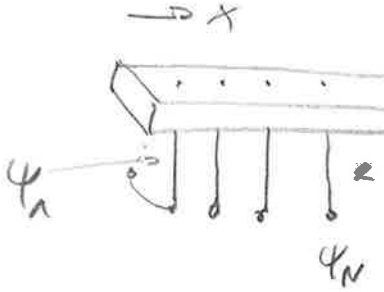
as $V = 4\tan^{-1}\left\{c\,\exp\left[-\frac{1}{\sqrt{\vartheta^2-1}}(y + \vartheta\tau)\right]\right\} \in [-2\pi, 2\pi]$

and $\vartheta^2 - 1 = \frac{(\alpha+1)^2}{(\alpha-1)^2} - 1 = \frac{4\alpha^2}{(\alpha^2-1)^2}$

- $\exists$ further solutions due to symmetries $\tau \to -\tau$, $y \to -y$ of Sine-Gordon

Real world model for the Sine-Gordon eq [Scott 1970]

- put pins in a rubber band ($\sim 100\,cm \times 6\,mm \times 2\,mm$)



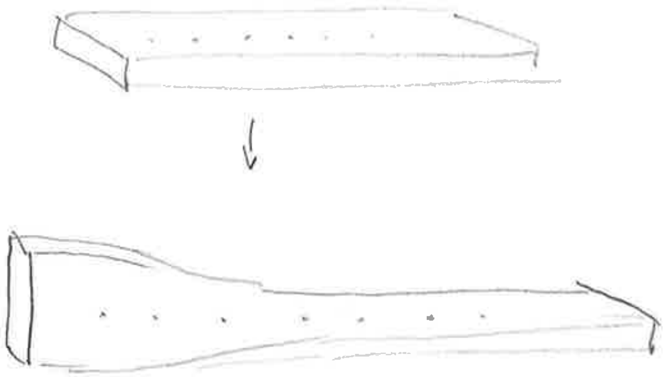
- equation of motion for a pendulum of length l

$$l\,\partial_t^2 \psi_1(t) = -g \sin \psi_1(t)$$

ψ deflection angle at fixed x

- equation of motion for torsion waves in elastic strip

$$\partial_x^2 \psi(x,t) = c^2 \partial_t^2 \psi(x,t)$$



$\Rightarrow$ chain of pendules $\psi_1(t), \dots \psi_N(t)$ are a discrete model for $\psi(x,t)$

- you may scatter torsion waves $\rightarrow$ soliton scattering

• example multisoliton solutions (multi-kink-solution)

⇝ apply several BT's as

* BT map solutions of non-lin. diff. eqs. onto each other, e.g. Sine-Gordon

$$\underset{\text{solution 1.}}{u} \xrightarrow[\text{param. } \alpha]{BT} \underset{\text{solution 2.}}{v}$$

* enchaining BT's leads to

$$u_0 \xrightarrow[\alpha_1]{} u_1 \xrightarrow[\alpha_2]{} u_{12}$$

α_1, α_2 BT parameters

or
$$u_0 \xrightarrow[\alpha_2]{} u_2 \xrightarrow[\alpha_1]{} u_{21}$$

Q: Does the order matter? A: NO: ∃ Theorem of interchangeability [Bianchi]

$$\boxed{u_{12} = u_{21}}$$

Consequences:

we have
$$\partial_+(u_0 - u_1) = 2\alpha_1 \sin\left(\frac{u_0+u_1}{2}\right)$$
$$\partial_-(u_0 + u_1) = -\frac{2}{\alpha_1} \sin\left(\frac{u_0-u_1}{2}\right)$$

⇒ compare
$$\partial_+(u_0-u_1) + \partial_+(u_1 - u_{12}) = \partial_+(u_0 - u_{12})$$
with
$$\partial_+(u_0-u_2) + \partial_+(u_2 - u_{21}) = \partial_+(u_0 - \overset{|||}{u_{21}})$$

$$\Rightarrow \quad 2\alpha_1 \sin\frac{u_0+u_1}{2} + 2\alpha_2 \sin\frac{u_1+u_{12}}{2} \overset{!}{=} 2\alpha_2 \sin\frac{u_0+u_2}{2} + 2\alpha_1 \sin\frac{u_2+u_{12}}{2}$$

$$\Leftrightarrow \quad 0 = \alpha_1\left(\sin\frac{u_2+u_{12}}{2} - \sin\frac{u_0+u_1}{2}\right) + \alpha_2\left(\sin\frac{u_0+u_2}{2} - \sin\frac{u_1+u_{12}}{2}\right)$$

- use identity from p. 104 $\quad \sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}$

$$\Rightarrow \quad 0 = \underbrace{2\cos\left(\frac{u_0+u_1+u_2+u_{12}}{4}\right)}_{\text{for generic } t \ \neq 0}\underbrace{\left(\alpha_1\sin\frac{u_2+u_{12}-u_0-u_1}{4} + \alpha_2\sin\frac{u_0+u_2-u_1-u_{12}}{4}\right)}_{\overset{!}{=}0}$$

for generic t $\neq 0 \quad \Rightarrow \quad \overset{!}{=} 0$

- use addition theorem $\quad \sin\alpha\pm\beta = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$

choose $\beta = \frac{u_{12}-u_0}{4}$, $\alpha = \frac{u_2-u_1}{4}$

$$\Rightarrow \quad 0 = \alpha_1\left(\sin\alpha\cos\beta + \sin\beta\cos\alpha\right) + \alpha_2\left(\sin\alpha\cos\beta - \sin\beta\cos\alpha\right)$$

$$= (\alpha_1+\alpha_2)\sin\left(\frac{u_2-u_1}{4}\right)\cos\left(\frac{u_{12}-u_0}{4}\right) + (\alpha_1-\alpha_2)\sin\left(\frac{u_{12}-u_0}{4}\right)\cos\left(\frac{u_2-u_1}{4}\right)$$

$\alpha_1 \neq \alpha_2$, $\cos\left(\frac{u_{12}-u_0}{4}\right) \neq 0 \neq \cos\left(\frac{u_2-u_1}{4}\right)$, divide by all

$$\Rightarrow \quad 0 = \frac{\alpha_1+\alpha_2}{\alpha_1-\alpha_2}\tan\left(\frac{u_2-u_1}{4}\right) + \tan\left(\frac{u_{12}-u_0}{4}\right)$$

$$\Leftrightarrow \quad \boxed{u_{12}(x^+,x^-) = u_0(x^+,x^-) + 4\tan^{-1}\left[\frac{\alpha_1+\alpha_2}{\alpha_2-\alpha_1}\tan\left(\frac{u_2-u_1}{4}\right)\right]}$$

- self consistent as symmetric under $u_1 \leftrightarrow u_2$, $\alpha_1 \leftrightarrow \alpha_2$

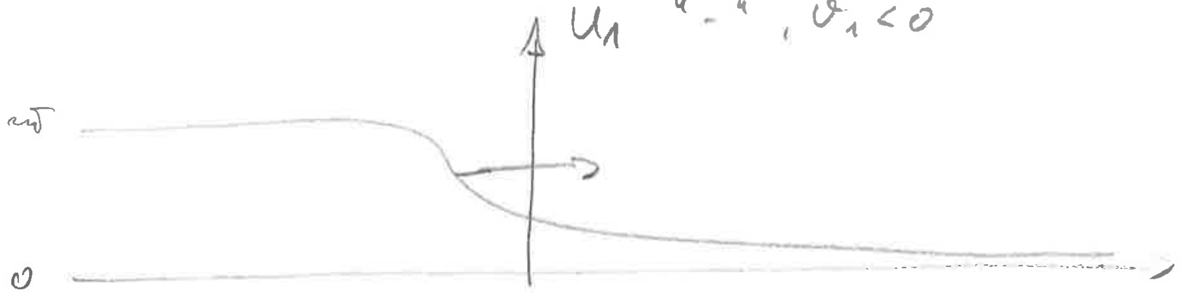
- algebraic construction, from "simpler" u_1, u_2 to more complicated u_{12} — may be iterated

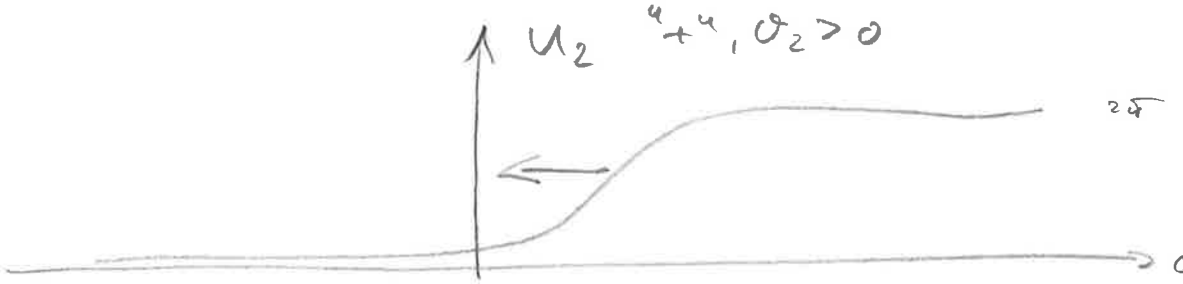
- in choosing $u_0 \equiv 0$ both u_1 and u_2 are 1 soliton solutions $\longrightarrow$ we will see that u_{12} $(= u_{21})$ is a 2-soliton solution :

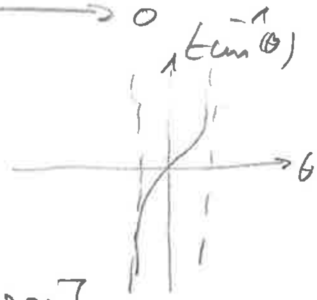
$$u_j(\tau, y) = 4 \tan^{-1}\left[c_j \exp\left[\pm \frac{1}{\sqrt{\vartheta_j^2 - 1}} (y + \vartheta_j \tau) \right]\right]$$

with $j = 1, 2$, $\vartheta_j = \dfrac{\alpha_j^2 + 1}{\alpha_j^2 - 1}$

for example we may choose



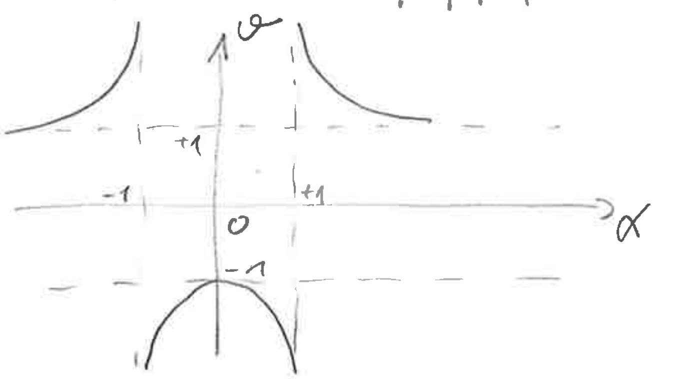




- for $u_0 \equiv 0$ we have

$$u_{12} = 4 \tan^{-1}\left[\underbrace{\left(\frac{\alpha_1 + \alpha_2}{\alpha_2 - \alpha_1}\right)} \underbrace{\tan\left(\frac{u_2 - u_1}{4}\right)} \right] \in [-2\pi, 2\pi]$$

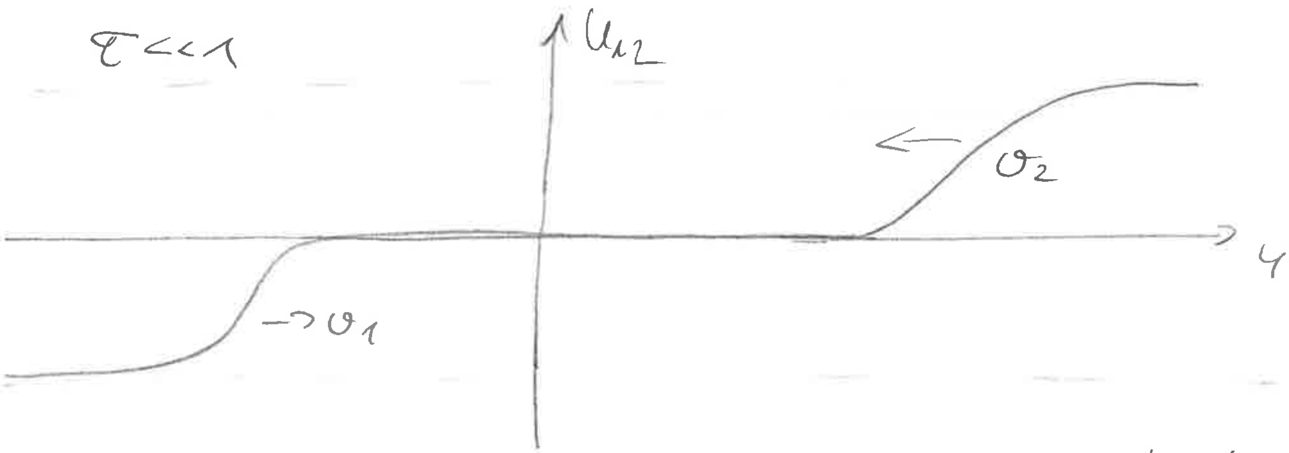
for $\vartheta_2 > 0 > \vartheta_1 \Rightarrow |\alpha_2| > |\alpha_1| \Rightarrow \; > 0 \qquad \in \mathbb{R}$



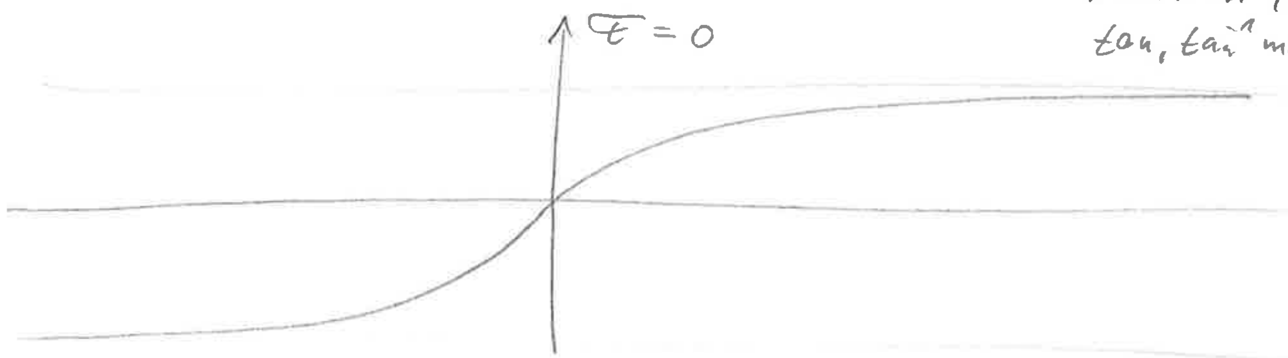
2 Soliton scattering

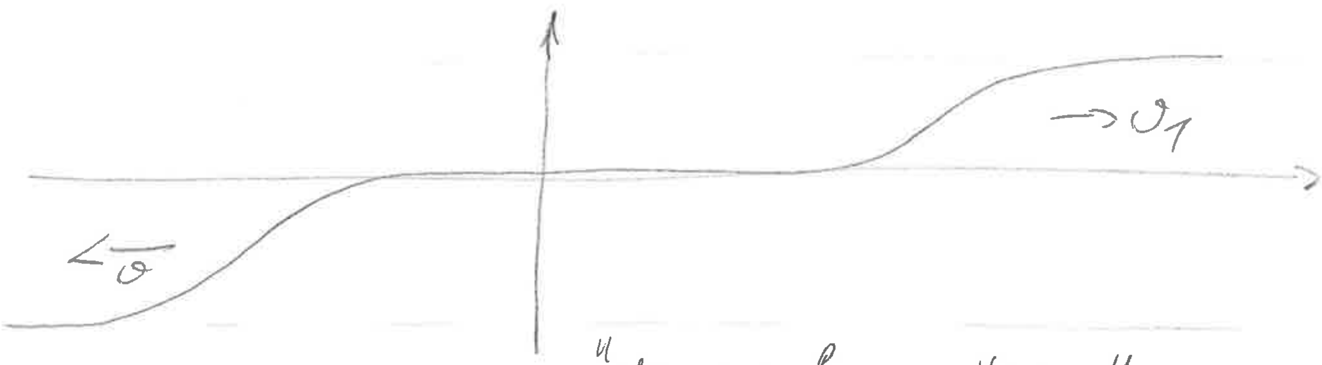
choose:
boundary conditions
for $\tau \ll 1$ and $\tau \gg 1$

	$y \to -\infty$	$y \to +\infty$
u_1	2π	0
u_2	0	2π
$\Rightarrow$ $u_2 - u_1$	-2π	$+2\pi$
$\tan\left(\frac{u_2-u_1}{4}\right)$	$-\infty$	$+\infty$
$4\tan^{-1}\left[(\,)\tan\frac{u_2-u_1}{4}\right]$	-2π	2π



+ continuity at 0,
tan, $\tan^{-1}$ monot.





"dispersionless scattering"