Bäcklund Transformation for the KdV eq.

- we already know the BT : the Miura trafo (or general, Riccati-) p. 52 and Exercise 5.3

KdV
$$\boxed{\partial_t u = u\,\partial_x u + \partial_x^3 u}$$

Miura trafo
$$u = -6\lambda + i\sqrt{6}\,\partial_x v + v^2 \qquad (*)$$

$\Rightarrow$ MKdV
$$\boxed{\partial_t v = (v^2 - 6\lambda)\,\partial_x v + \partial_x^3 v}$$

- read as a BT
$$\partial_x v = F(u, v)$$
$$\partial_t v = G(u, v)$$
:

$(*) \Leftrightarrow$
$$\boxed{\partial_x v = \frac{-i}{\sqrt{6}}\left(u + 6\lambda - v^2\right)}$$
use to eliminate all $\partial_x^n v$ in MKdV

$$\partial_x^3 v = \frac{-i}{\sqrt{6}}\left(\partial_x^2 u - 2\partial_x (v\,\partial_x v)\right)$$
$$= \frac{-i}{\sqrt{6}}\left(\partial_x^2 u - 2(\partial_x v)^2 - 2v\,\partial_x^2 v\right)$$

$$\partial_x^2 v = -\frac{i}{\sqrt{6}}\left(\partial_x u - 2v\,\partial_x v\right)$$

$$\Rightarrow \partial_t v = \frac{-i}{\sqrt{6}}\Big[(v^2 - 6\lambda)(u + 6\lambda - v^2) + \partial_x^2 u - 2\left(\frac{-1}{6}\right)(u + 6\lambda - v^2)^2 - 2v\frac{-i}{\sqrt{6}}\left(\partial_x u - 2v\frac{-i}{\sqrt{6}}(u + 6\lambda - v^2)\right)\Big]$$
$$= \frac{-i}{\sqrt{6}}\Big[\partial_x^2 u + \left(v^2 - 6\lambda + \frac{1}{3}(u + 6\lambda - v^2) + \frac{2}{3}v^2\right)(u + 6\lambda - v^2)\Big]$$
$$+ \frac{1}{3}v\,\partial_x u$$

$$\Longleftrightarrow \boxed{\partial_t v = \frac{-i}{\sqrt{6}}\left[\partial_x^2 u + \left(\tfrac{4}{3}v^2 - 4\lambda^2 + \tfrac{u}{3}\right)(u + 6\lambda - v^2)\right] + \frac{v}{3}\partial_x u}$$

- this BT $u \Longleftrightarrow v$ maps solutions of the KdV onto solutions of the MKdV

But: if we want to construct multi-soliton solutions of the KdV in analogy to Sine-Gordon we need to find a BT mapping KdV to KdV!

Note: a property of MKdV is: if v is a solution also $-v$ is a solution of MKdV

(not true for KdV!)

Trick: find such a BT mapping the same eq onto itself

for $\int^x dx' u(x',t) \equiv w(x,t) \Longleftrightarrow u(x,t) = \partial_x w(x,t)$

$\Rightarrow$ KdV for $\partial_x w$: $\partial_t \partial_x w = \underbrace{\partial_x w \, \partial_x^2 w}_{\frac{1}{2}\partial_x(\partial_x w)^2} + \partial_x^4 w$

integrate $\int^{x'} dx$:

$$\Rightarrow \boxed{\partial_t w(x',t) = \frac{1}{2}\left(\partial_{x'} w(x',t)\right)^2 + \partial_{x'}^3 w(x',t)} \quad \Delta$$

seek a BT $w \to \tilde{w}$ that satisfies the same eq.

- Idea: Miura trafo maps sol $v \to u = \partial_x w$

  consider map sol $-v \to \tilde{u} \equiv \partial_x \tilde{w}$

$$\Leftrightarrow \qquad \partial_x v = \frac{-i}{\sqrt{6}}(u + 6\lambda - v^2)$$

$$-\partial_x v = \frac{-i}{\sqrt{6}}(\tilde{u} + 6\lambda - v^2)$$

difference: $\partial_x v = \frac{-i}{2\sqrt{6}}(u - \tilde{u}) \overset{\int}{\Longrightarrow} v = \frac{-i}{2\sqrt{6}}(w - \tilde{w})$

sum: $\frac{1}{2}(u + \tilde{u}) = v^2 - 6\lambda$

$$\Rightarrow \boxed{\partial_x(w + \tilde{w}) = -12\lambda + \cancel{2}\frac{(-1)}{\cancel{4}_{2} \cdot 6}(w - \tilde{w})^2}$$

1. BT eq
parameter λ

insert above into MKdV:

$$\partial_t v = \frac{1}{2}(u + \tilde{u})\frac{-i}{2\sqrt{6}}(u - \tilde{u}) + \partial_x^2\left(\frac{-i}{2\sqrt{6}}\right)(u - \tilde{u})$$

$$\Leftrightarrow \boxed{\partial_t(w - \tilde{w}) = \frac{1}{2}\left((\partial_x w)^2 - (\partial_x \tilde{w})^2\right) + \partial_x^3 w - \partial_x^3 \tilde{w}} \qquad 2.$$

$\Rightarrow$ both $\bar{w}$ and $\tilde{w}$ satisfy the same eq. 1 !!

(we could make 2. a proper BT eq by eliminating all ∂_x derivatives on the rhs using eq. 1.)

1 Soliton solution:

start with $w \equiv 0 \xrightarrow[\lambda]{BT} \tilde{w}$:

$$\partial_x \tilde{w} = -12\lambda - \frac{1}{12}\tilde{w}^2$$

and $\partial_t \tilde{w} = \frac{1}{2}(\partial_x \tilde{w})^2 + \partial_x^3 \tilde{w}$

We can deduce:

$$\partial_x^2 \tilde{\omega} = -\frac{1}{6}\tilde{\omega}\,\partial_x\tilde{\omega} = +\frac{1}{6}\tilde{\omega}\left(12\lambda + \frac{1}{12}\tilde{\omega}^2\right) = 2\tilde{\omega}\lambda + \frac{1}{6\cdot 12}\tilde{\omega}^3$$

$$\Rightarrow \partial_x^3\tilde{\omega} = 2\,\partial_x\tilde{\omega}\,\lambda + \frac{3}{6\cdot 12}\tilde{\omega}^2\partial_x\tilde{\omega} = \left(2\lambda + \frac{1}{24}\tilde{\omega}^2\right)\left(-12\lambda - \frac{1}{12}\tilde{\omega}^2\right)$$

$$\Rightarrow \partial_t\tilde{\omega} = \frac{1}{2}\left(12\lambda + \frac{\tilde{\omega}^2}{12}\right)^2 - \left(2\lambda + \frac{1}{24}\tilde{\omega}^2\right)\left(12\lambda + \frac{1}{12}\tilde{\omega}^2\right)$$

$$= \frac{(12)^2\lambda^2}{2} + \lambda\tilde{\omega}^2 + \frac{1}{2}\frac{\tilde{\omega}^4}{(12)^2} - 24\lambda^2 - \frac{2}{12}\lambda\tilde{\omega}^2 - \frac{1}{2}\lambda\tilde{\omega}^2 - \frac{\tilde{\omega}^4}{2\cdot(12)^2}$$

$$= 2\cdot 24\lambda^2 + \frac{1}{3}\lambda\tilde{\omega}^2 = -4\lambda\,\partial_x\tilde{\omega}$$

$$\Leftrightarrow \quad \underline{(\partial_t + 4\lambda\partial_x)\,\tilde{\omega}(x,t) = 0}$$

$$\Rightarrow \quad \tilde{\omega} = \tilde{\omega}(x - 4\lambda t)$$

1 soliton solution propagates with speed $v = -4\lambda$ (time evol.!)

- otherwise the solution for $\tilde{\omega}$ is not simpler than solving KdV
- the construction of multi-soliton solutions <u>IS</u> simpler using BT:

$$\omega_0 \xrightarrow[\lambda_1]{BT} \omega_1 \xrightarrow[\lambda_2]{BT} \omega_{12}$$

$$\omega_0 \xrightarrow[\lambda_2]{BT} \omega_2 \xrightarrow[\lambda_1]{BT} \omega_{21}$$

Bianchi Th^m: $\omega_{21} = \omega_{12}$. Starting with $\omega_0 \equiv 0$ we have that ω_1, ω_2 are 1 soliton solutions $\Rightarrow$ ω_{21} is a 2 soliton one

• in analogy to Sine-Gordon compare

$$\partial_x(w_0+w_1) - \partial_x(w_1+w_{12}) = \partial_x(w_0-w_{12})$$

and $$\partial_x(w_0+w_2) - \partial_x(w_2+\overset{\shortparallel}{w_{21}}) = \partial_x(w_0-w_{12}) \quad \text{!}$$

$$\Longleftrightarrow \; -12(\lambda_1-\lambda_2) - \frac{1}{12}\left[(w_0-w_1)^2-(w_1-w_{12})^2\right]$$

$$\overset{!}{=} -12(\lambda_2-\lambda_1) - \frac{1}{12}\left[(w_0-w_2)^2-(w_2-w_{12})^2\right]$$

$$\Longleftrightarrow \; 2\cdot 12^2(\lambda_2-\lambda_1) = \big[w_0^2-2w_0w_1+w_1^2-w_1^2+2w_1w_{12}-w_{12}^2$$
$$-w_0^2+2w_0w_2-w_2^2+w_2^2-2w_2w_{12}+w_{12}^2\big]$$

$$= 2(w_0-w_{12})(w_2-w_1)$$

$$\Longleftrightarrow \; \boxed{w_{12}(x,t) = w_0(x,t) - 144\,\frac{\lambda_2-\lambda_1}{w_2(x,t)-w_1(x,t)}}$$

* 2 soliton solution from 2 1 solitons w_1, w_2 for $w_0 \equiv 0$
* >2 " by iteration!

∘ recall that 1 soliton solution (eg. p. 35 $c=-\lambda\cdot 4$)

$$u(x,t) = \frac{-12\lambda}{\cosh^2\left[\sqrt{-\lambda}\,(x-4\lambda t)\right]} = \partial_x w(x,t)$$

$$\Longrightarrow \; w_j(x,t) = 12\sqrt{-\lambda_j}\,\tanh\left[\sqrt{-\lambda_j}\,(x-4\lambda_j t)\right], \quad j=1,2$$

$$\Longrightarrow \; w_{12}(x,t), \text{ and by differentiation } u_{12}(x,t)$$

• one can show that asymptot. we have 1 soliton solution: $\xi_j = \sqrt{-\lambda_j}\,(x-4\lambda_j t)$

$$u_{12} \xrightarrow[|\xi_{1,2}|\to+\infty]{} \frac{-12\lambda_{1,2}}{\cosh^2\left[\xi_{1,2}\mp\gamma\right]} \quad \text{with } \gamma = \tanh^{-1}\sqrt{\frac{\lambda_1}{\lambda_2}}$$

More than 2 Solitons: iteration

start with 1 soliton $w_1 \xrightarrow{\lambda_2} w_{12} \xrightarrow{\lambda_3} w_{123}$

$\xrightarrow{\lambda_3} w_{13} \xrightarrow{\lambda_2} w_{123}$

$$\Rightarrow \quad w_{123} = w_1 - 144 \frac{\lambda_3 - \lambda_2}{w_{13} - w_{12}}$$

inserting
$$w_{12} = w_0 - 144 \frac{\lambda_2 - \lambda_1}{w_2 - w_1}$$
$$w_{13} = w_0 - 144 \frac{\lambda_3 - \lambda_1}{w_3 - w_1}$$

we obtain a 3 soliton solution in terms of 1 soliton ones (check)

$$w_{123} = \frac{\lambda_1 w_1 (w_2 - w_3) + \lambda_2 w_2 (w_3 - w_1) + \lambda_3 w_3 (w_1 - w_2)}{\lambda_1 (w_2 - w_3) + \lambda_2 (w_3 - w_1) + \lambda_3 (w_1 - w_2)}$$

which has explicit permutation symmetry in 1, 2, 3

Higher dimensional generalisation of KdV:

* Kadomtsev - Petviashvili equation in 1+2 dimensions

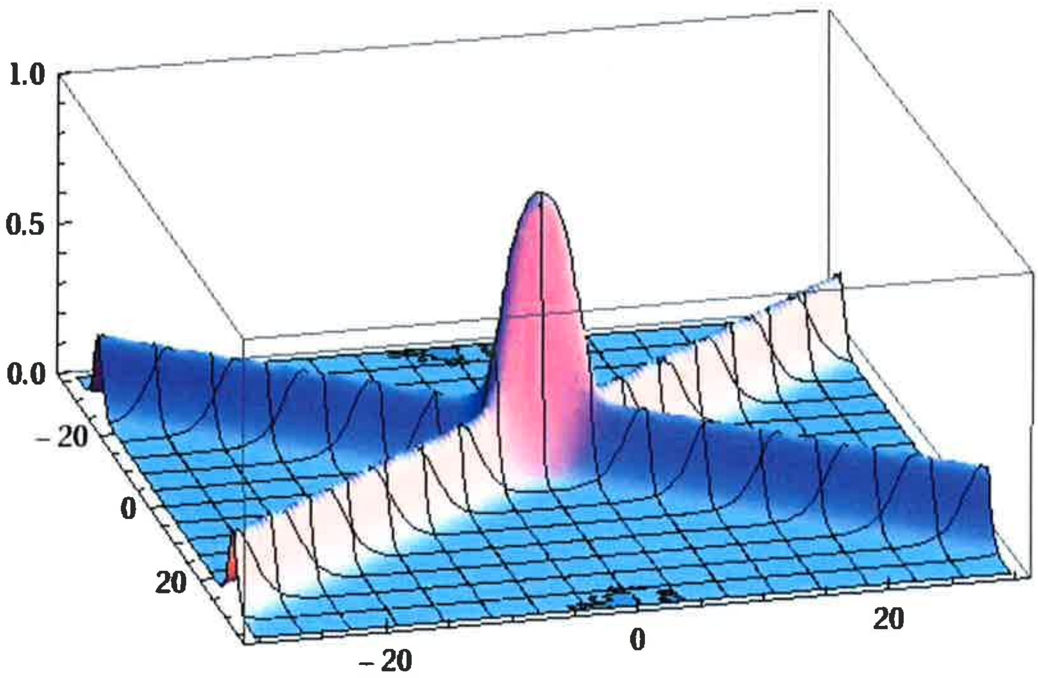
KPII
$$(\underbrace{u_t + 6uu_x + u_{xxx}}_{\text{KdV}})_x + 3\sigma^2 u_{yy} = 0$$

$\Rightarrow$ any solution of KdV is a solution of KP

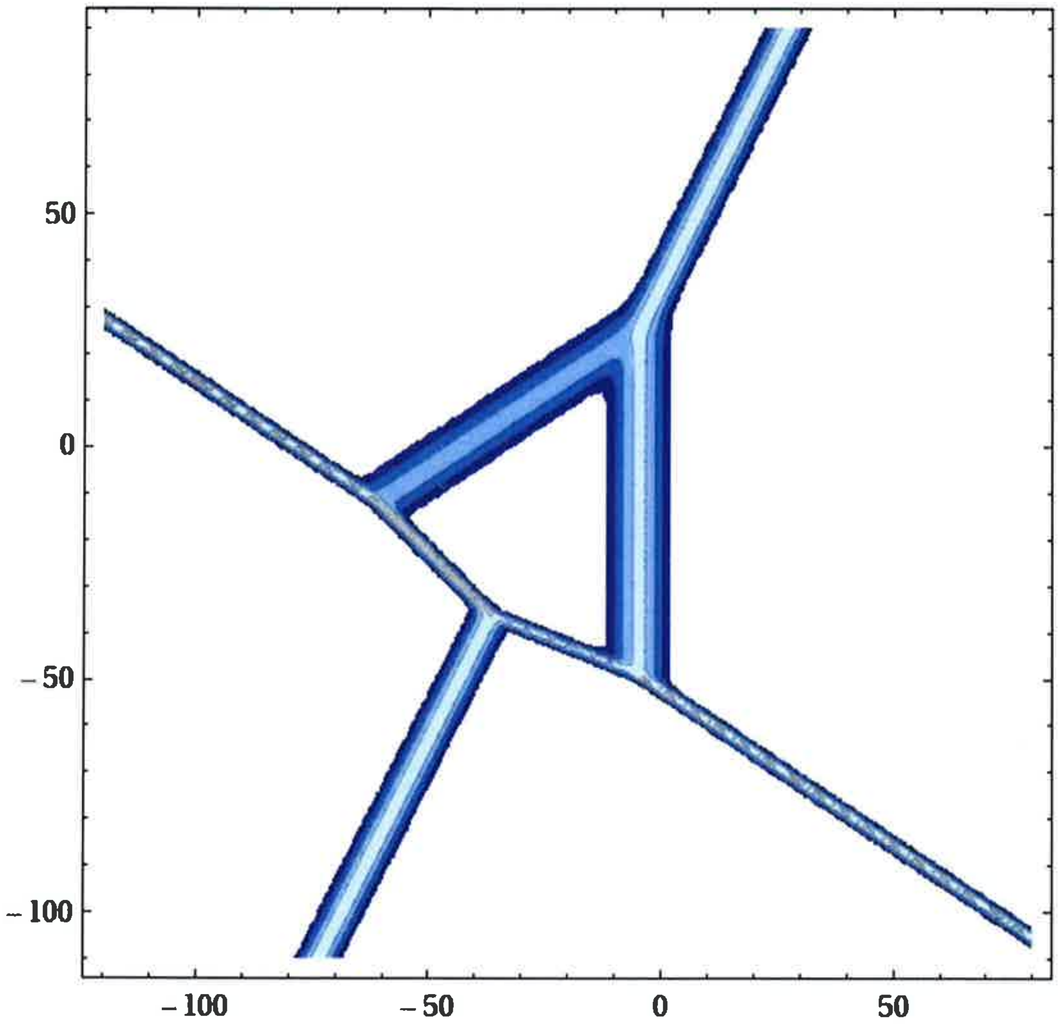
Example: line soliton $u(x,y,t) = \frac{a^2}{2} \frac{1}{\cosh^2\left[\frac{a}{2}(x - by - \omega\frac{t}{a} - x_0)\right]}$

for more see scholarpedia $\rightarrow$ pictures

An ordinary 2-soliton solution of KPII



A resonant 2 soliton solution of KP II



A resonant interaction between 3 line soliton solutions of KPII