The Toda Lattice [M. Toda, 1967]

- System with N = finitely many degrees of freedom (unlike KdV: field)

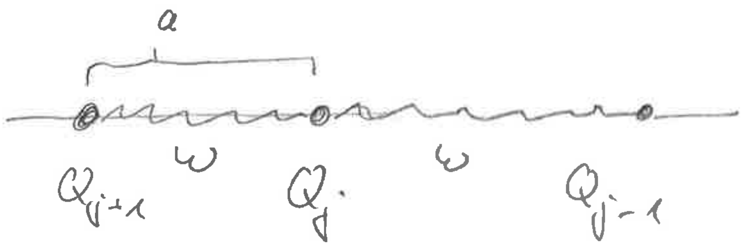
chain of N particles with exponential interaction

$$\boxed{\begin{aligned} \dot{Q}_i &= P_i \\ \dot{P}_j &= e^{-(Q_j - Q_{j-1})} - e^{-(Q_{j+1} - Q_j)} \end{aligned}} \qquad \begin{aligned} i &= 1, 2, \dots, N \\ j &= 1, \dots, N \end{aligned}$$

where we define $Q_0 \equiv -\infty = -Q_{N+1}$

(similar to field eqs. Liouville $(\partial_t^2 - \partial_x^2) u = e^u$ or
sinh-Gordon $(\partial_t^2 - \partial_x^2) u = \frac{1}{2}(e^u - e^{-u})$)

- the Toda lattice generalises a chain of N harmonic oscillators



a: spring in rest
ω: spring const
equal ∀j

$$\Rightarrow m \ddot{Q}_j = \omega(Q_{j+1} - Q_j - \cancel{a}) - \omega(Q_j - Q_{j-1} - \cancel{a})$$

linearised form of the force $\omega(1 - e^{-(Q_{j+1} - Q_j)})$

- we may remove $\frac{m}{\omega}$ by rescaling t,
(or ω in Toda, set $\equiv 1$ above already)

- We will show that the Toda Lattice has 2 PB and construct its N conserved charges

→ will show general 2 PB $\Leftrightarrow$ $\exists$ Lax pair

- new feature : Toda has a group structure & $\exists$ interesting examples realising the Toda Lattice structure: random matrix models

reminder Poisson Bracket (for discrete variables)

for canonical PB

$$i = 1, \dots, N \qquad \dot{q}_i = \{q_i, H\} = \frac{\partial}{\partial p_i} H$$

$$\dot{p}_i = \{p_i, H\} = -\frac{\partial}{\partial q_i} H$$

- ∂_t, H Hamiltonian (not explicitly t-dep)

q_i, p_i indep variables

equivalent form : symplectic structure (→ fields)

$$\mu = 1, \dots, 2N \qquad \dot{y}^\mu = \{y^\mu, H\} = f^{\mu\nu} \underbrace{\frac{\partial}{\partial y^\nu}}_{\partial_\nu} H$$

$y^\mu = (q^1 \dots q^N, p^1, \dots, p^N)$

with canonical PB $f^{\mu\nu} = \epsilon^{\mu\nu} = \begin{pmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{pmatrix}$

- in general $\{A, B\} = \partial_\mu A \cdot f^{\mu\nu} \partial_\nu B$

$$\dot{A} = \{A, H\}$$

∃ 2 Poisson brackets for Toda

1. canonical PB :

$$H_0(Q,P) = \sum_{i=1}^{N}\left(\frac{1}{2}P_i^2 + e^{-(Q_{i+1}-Q_i)}\right)$$

$$\partial_{P_i} H_0 = P_i = \dot{Q}_i \qquad i=1,\dots,N$$

$$\partial_{Q_i} H_0 = e^{-(Q_{i+1}-Q_i)} - e^{-(Q_i - Q_{i-1})} = -\dot{P}_i$$

$$\Rightarrow \quad f^{\mu\nu} = \epsilon^{\mu\nu} \quad \text{in} \quad \dot{y}^\mu = f^{\mu\nu}\partial_\nu H_0$$

2. PB :

$$H_1(Q,P) \equiv \sum_{i=1}^{N}\left(\frac{1}{3}P_i^3 + (P_i + P_{i+1})e^{-(Q_{i+1}-Q_i)}\right)$$

seek $F^{\mu\nu}$ s.th. $\dot{y}^\mu = F^{\mu\nu}\partial_\nu H_1$ gives the Toda eqs.

• simpler : find $(F^{\mu\nu})^{-1} = F_{\mu\nu}$

$$\underline{\partial_g H_1} = F_{g\mu}\dot{y}^\mu \overset{1.\,PB}{=} \underline{F_{g\mu} f^{\mu\nu}\partial_\nu H_0}$$

define $\boxed{S_g{}^\nu \equiv F_{g\mu} f^{\mu\nu}}$ s.th. $\partial_g H_1 = S_g{}^\nu \partial_\nu H_0$

• later we'll show for general F, f that S gives one of the Lax op. and at the same time gives the conserved quantities

claim :

$$F_{\mu\nu} = \begin{pmatrix} A & -B \\ B & e \end{pmatrix} \quad \text{with matrices} \quad \begin{aligned} A_{ij} &= \delta_{i+1,j}\, e^{-(Q_{i+1}-Q_i)} - \delta_{i,j+1}\, e^{-(Q_{j+1}-Q_j)} \\ B_{ij} &= \delta_{ij} P_i \\ e_{ij} &= \epsilon(j-i) = \begin{cases} 1 & j>i \\ 0 & j=i \\ -1 & j<i \end{cases} \end{aligned}$$

we have on the l.h.s.

$$(i)\quad \partial_{Q_i} H_1 = -(P_{i-1}+P_i)\,e^{-(Q_i-Q_{i-1})} + (P_i+P_{i+1})\,e^{-(Q_{i+1}-Q_i)}$$

$$(ii)\quad \partial_{P_i} H_1 = P_i^2 + e^{-(Q_i-Q_{i-1})} + e^{-(Q_{i+1}-Q_i)}$$

on the r.h.s:

$$F_{g\mu} f^{\mu\nu} \partial_\nu H_0 = \begin{pmatrix} A & -B \\ B & e \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} \partial_Q H_0 \\ \partial_P H_0 \end{pmatrix} = \begin{pmatrix} A & -B \\ B & e \end{pmatrix}\begin{pmatrix} \partial_P H_0 \\ -\partial_Q H_0 \end{pmatrix}$$

1st row (i):

$$A\partial_P H_0 + B\partial_Q H_0 = \left(\delta_{i+1,j}\, e^{-(Q_{i+1}-Q_i)} - \delta_{i,j+1}\, e^{-(Q_{j+1}-Q_j)}\right) P_j$$
$$+ \delta_{ij} P_i \left(e^{-(Q_{j+1}-Q_j)} - e^{-(Q_j-Q_{j-1})}\right)$$
$$= (P_{i+1}+P_i)\,e^{-(Q_{i+1}-Q_i)} - (P_{i-1}+P_i)\,e^{-(Q_i-Q_{i-1})} \quad \checkmark$$

2nd row (ii):

$$B\partial_P H_0 - e\partial_Q H_0 = \delta_{ij} P_i P_j - \epsilon(j-i)\left(e^{-(Q_{j+1}-Q_j)} - e^{-(Q_j-Q_{j-1})}\right)$$
$$= P_i^2 - \sum_{j=i+1}^{N}\left(e^{-(Q_{j+1}-Q_j)} - e^{-(Q_j-Q_{j-1})}\right)$$
$$+ \sum_{j=1}^{i-1}\left(e^{-(Q_{j+1}-Q_j)} - e^{-(Q_j-Q_{j-1})}\right)$$

use $Q_0 = -\infty$, $Q_N = +\infty$ and telescopic sum

$$= P_i^2 - (-)e^{-(Q_{i+1}-Q_i)} + e^{-(Q_i-Q_{i-1})}$$

$\checkmark$

Show now: $\exists$ 2 PB $\Leftrightarrow$ $\exists$ Lax-Pair :

- as a first result we need :

given both Lagrangian L and Hamiltonian H we can compute $f_{\mu\nu}$

Hamiltonian $H_0 = H_0(\theta, y)$

Lagrangian $L_0 = \theta^0_\mu(y)\,\dot{y}^\mu - H_0(y)$; $\theta^0_\mu(y) = \frac{\partial}{\partial \dot{y}^\mu} L_0(\dot{y}, y)$

conjugate momenta to y

e.o.m.:

for H_0 : $\underline{\dot{y}^\mu = f^{\mu\nu}\partial_\nu H_0}$ $\Leftrightarrow$ $f_{\nu\mu}\dot{y}^\mu = \partial_\nu H_0$

for L_0 : $\underline{0 = \left(\partial_t \frac{\partial}{\partial \dot{y}^\mu} - \frac{\partial}{\partial y^\mu}\right) L_0(\dot{y}, y)}$, $\partial_\mu = \frac{\partial}{\partial y^\mu}$

using the def: $0 = \partial_t\, \theta^0_\mu(y) - \partial_\mu\left(\theta^0_\nu \dot{y}^\nu - H_0(y)\right)$

$$= \partial_t y^\nu \partial_\nu \theta^0_\mu - \left(\partial_\mu \theta^0_\nu\right)\dot{y}^\nu - \theta^0_\nu \underbrace{\partial_\mu \dot{y}^\nu}_{=0 \text{ as } y, \dot{y} \text{ indep for } L} + \partial_\mu H_0$$

$$\Leftrightarrow \quad \partial_\mu H_0 = \partial_\mu \theta^0_\nu - \partial_\nu \theta^0_\mu \;\dot{y}^\nu$$

$$\Rightarrow \quad \boxed{f_{\mu\nu} = \partial_\mu \theta^0_\nu - \partial_\nu \theta^0_\mu}$$

example canonical P.B.:

$$L_0 = \theta^0_\mu(y)\dot{y}^\mu - H_0(y) \overset{\text{write out sum}}{=} \sum_{i=1}^N \frac{1}{2}\left(P_i \dot{Q}_i - Q_i \dot{P}_i\right) - H_0(P, Q)$$

$$= \sum_{i=1}^N \frac{1}{2}\left(y_{i+N}\dot{y}_i - y_i \dot{y}_{i+N}\right) - H_0$$

$$= \underbrace{-\frac{1}{2} y^3 \epsilon_{3\mu}}\, \dot{y}^\mu - H_0$$

$$\Leftarrow \quad \theta^0_\mu(y) \Rightarrow f_{\mu\nu} = -\frac{1}{2}\partial_\mu y^3 \epsilon_{3\nu} + \frac{1}{2}\partial_\nu y^3 \epsilon_{3\mu} = \epsilon_{\nu\mu}$$

- the same holds for the second PB:

$$\dot{y}^{\nu} = f^{\nu\mu}\partial_\mu H_0 \quad , \quad (f^{\mu\nu})^{-1} = f_{\mu\nu} = \partial_\mu \theta^0_\nu - \partial_\nu \theta^0_\mu$$

$$\dot{y}^{\nu} = F^{\nu\mu}\partial_\mu H_1 \quad , \quad (F^{\mu\nu})^{-1} \; F_{\mu\nu} = \partial_\mu \theta^1_\nu - \partial_\nu \theta^1_\mu$$

<u>Claim</u>: $S_\mu{}^\nu = F_{\mu\lambda} f^{\lambda\nu}$ satisfies the following

Lax-eq. $\boxed{\partial_t S_\mu{}^\nu = [S, U]_\mu{}^\nu} = S_\mu{}^\lambda U_\lambda{}^\nu - U_\mu{}^\lambda S_\lambda{}^\nu$, with 'U below

<u>recall</u>: $L_0 = L_0(\dot{y}, y)$ indep variables

$H_0 = H_0(\theta^0(y), y)$ " variables

- H_0 smooth $\Rightarrow \partial_\mu \partial_\nu H_0 = \partial_\nu \partial_\mu H_0 \quad , \quad \partial_\mu H_0 = \frac{\partial}{\partial y_\mu} H_0(\theta^0(y), y)$ diff both

- <u>e.o.m</u> $f_{\mu\nu}\dot{y}^\nu = \partial_\mu H_0$ inserted into above, with $\dot{y} = \dot{y}(\theta(y), y)$

$$\Rightarrow \underline{0 = \partial_\mu(f_{\nu\rho}\dot{y}^\rho) - \partial_\nu(f_{\mu\rho}\dot{y}^\rho)} = (\partial_\mu f_{\nu\rho} - \partial_\nu f_{\mu\rho})\dot{y}^\rho + (f_{\nu\rho}\partial_\mu - f_{\mu\rho}\partial_\nu)\dot{y}^\rho$$

writing $f_{\mu\nu} = f_{\mu\nu}(y)$ we have

$$\partial_t f_{\mu\nu} = (\partial_t y^\rho)\,\partial_\rho f_{\mu\nu} \overset{\text{Bianchi Id}}{=} \dot{y}^\rho(-\partial_\mu f_{\nu\rho} - \partial_\nu \underset{\text{anti-sym}}{f_{\rho\mu}})$$

$$\overset{\text{insert above}}{=} f_{\nu\rho}\partial_\mu \dot{y}^\rho - f_{\rho\mu}\partial_\nu \dot{y}^\rho$$

defining $U_\mu{}^\nu \equiv \partial_\mu \dot{y}^\nu = \partial_\mu(f^{\nu\rho}\partial_\rho H_0) \neq 0$

$$\Longleftrightarrow \boxed{\partial_t f_{\mu\nu} = f_{\nu\rho}U_\mu{}^\rho - f_{\rho\mu}U_\nu{}^\rho = -U_\mu{}^\rho f_{\rho\nu} + U_\nu{}^\rho f_{\rho\mu}}$$

ditto $\boxed{\partial_t F_{\mu\nu} = -U_\mu{}^\rho F_{\rho\nu} + U_\nu{}^\rho F_{\rho\mu}}$

$$\Rightarrow \underline{\partial_t S_\mu{}^\nu} = (\partial_t F_{\mu\lambda}) f^{\lambda\nu} + F_{\mu\lambda}\partial_t f^{\lambda\nu}$$

$$= -U_\mu{}^\varrho F_{\varrho\lambda} f^{\lambda\nu} + U_\lambda{}^\varrho F_{\varrho\mu} f^{\lambda\nu} - F_{\mu\lambda} f^{\lambda\mu'}(\partial_t f_{\mu'\varrho}) f^{\varrho\nu}$$

as $\partial_t\left(f_{\mu\varrho} f^{\varrho\nu} = \delta_\mu{}^\nu\right) = 0 \iff (\partial_t f_{\mu\varrho}) f^{\varrho\nu} = -f_{\mu\varrho}\partial_t f^{\varrho\nu}$

$$\overset{\cdot f^{\lambda\mu}}{\iff} \partial_t f^{\lambda\nu} = -f^{\lambda\mu}(\partial_t f_{\mu\varrho}) f^{\varrho\nu}$$

$$= -U_\mu{}^\varrho S_\varrho{}^\nu + U_\lambda{}^\varrho F_{\varrho\mu} f^{\lambda\nu} + F_{\mu\lambda} f^{\lambda\mu'} U_{\mu'}{}^\sigma f_{\sigma\varrho} f^{\varrho\nu} - F_{\mu\lambda} f^{\lambda\mu'} U_\varrho{}^\sigma f_{\sigma\mu'} f^{\varrho\nu}$$

$$= \underline{-U_\mu{}^\varrho S_\varrho{}^\nu} + U_\lambda{}^\varrho F_{\varrho\mu} f^{\lambda\nu} + \underline{S_\mu{}^\varrho U_\varrho{}^\nu} + F_{\mu\sigma} U_\varrho{}^\sigma f^{\varrho\nu}$$

<u>Note:</u> instead of the Lax pair $\{S_\mu{}^\nu = F_{\mu\lambda} f^{\lambda\nu},\ U_\mu{}^\nu = \partial_\mu(f^{\nu\varrho}\partial_\varrho H_0)\}$

we may also use $\underline{U_\mu}{}^\nu = \partial_\mu \dot{y}^\nu = \partial_\mu\left(F^{\nu\varrho}\partial_\varrho H_1\right)$

<u>Conserved quantities:</u>

$$\boxed{\begin{aligned} K_0 &\equiv \ln|\det S| \\ K_n &\equiv \frac{1}{n}\operatorname{Tr}(S^n) \end{aligned}} \quad n\in\mathbb{N}$$

as

$$\partial_t K_0 = \partial_t \ln|\det S| = \partial_t \operatorname{Tr}\ln|S| \overset{\operatorname{Tr}S\neq 0}{=} \frac{1}{\operatorname{Tr}S}\partial_t\operatorname{Tr}S$$

$$= \frac{1}{\operatorname{Tr}S}\operatorname{Tr}[S,U] = \frac{1}{\operatorname{Tr}S}\operatorname{Tr}(SU - US) = 0$$

$$\partial_t K_n = \frac{1}{n}\sum_{i=1}^{n}\operatorname{Tr}\left(S^{i-1}(\partial_t S)S^{n-i}\right) = \frac{1}{n}\sum_{i=1}^{n}\operatorname{Tr}\left(S^{i-1}[S,U]S^{n-i}\right) = 0$$

due to the cyclicity of Tr

Example Toda-Lattice:

construct K_n based on $S_\mu{}^\nu = F_{\mu s} f^{s\nu} = \begin{pmatrix} A & -B \\ B & e \end{pmatrix}\begin{pmatrix} 0 & 1\!\!1 \\ -1\!\!1 & 0 \end{pmatrix} = \begin{pmatrix} B & A \\ -e & B \end{pmatrix}$

- $K_1 = \mathrm{Tr}\, S = 2\,\mathrm{Tr}\, B = 2 \sum_{i=1}^N P_i$ total momentum conservation

$$K_2 = \frac{1}{2} \mathrm{Tr}\, S^2 = \frac{1}{2} \mathrm{Tr}\left(2B^2 - (Ae + eA)\right)$$

$$= \sum_{i=1}^N P_i^2 - 2 \left(\delta_{i+1,j}\, e^{-(Q_{i+1}-Q_i)} - \delta_{i,j+1}\, e^{-(Q_{j+1}-Q_j)}\right) \epsilon(i-j)$$

$$= \sum_{j=1}^N P_j^2 - 2 e^{-(Q_{i+1}-Q_i)} = 2 H_0(P,Q)$$

Similarly $K_3 = \frac{1}{3} \mathrm{Tr}\, S^3 = \ldots = H_1(P,Q)$ exercise

One can show $K_n = \frac{1}{n} \sum_{i=1}^N P_i^n + \ldots$

- because the $P_{j=1,\ldots,N}$ are linearly indep. the first $K_{n=1,\ldots,N}$ are linearly indep too. These N conserved quantities are sufficient to make the system of N coupled eqs. integrable.

Note: it is possible to do the transition from discrete to continuous variables in this formalism, in order to derive the Lax-pair

p. 20-21: $F^{\mu\nu} \to \partial_x\, \delta(x-y)$

$f^{\mu\nu} \to \left(\partial_x^3 + \frac{1}{3}\partial_x u + u \partial_x\right) \delta(x-y)$

we then have $F^{-1} = G(x-y) = \theta(x-y) - \frac{1}{2}$, but no closed form for f^{-1}

$S = \partial_x^2 + \frac{2}{3} u + \frac{1}{3} (\partial_x u) \partial_x^{-1}$ leads to KdV

- group structure of the Toda-Lattice

- using conservation of total momentum we may change to centre of mass coord $q_a = Q_{a+1} - Q_a$, $a = 1, \dots, N-1$

e.o.m.

$$\ddot{Q}_1 = \cancel{e^{-(Q_1 - Q_0)}} - e^{-(Q_2 - Q_1)} = -e^{-q_1}$$

$$\ddot{Q}_i = e^{-(Q_i - Q_{i-1})} - e^{-(Q_{i+1} - Q_i)} = e^{-q_{i-1}} - e^{-q_i}$$

$$\ddot{Q}_N = e^{-(Q_N - Q_{N-1})} \cancel{- e^{-(Q_{N+1} - Q_N)}} = e^{-q_{N-1}}$$

$$\Rightarrow \quad \ddot{q}_1 = \ddot{Q}_2 - \ddot{Q}_1 = 2e^{-q_1} - e^{-q_2}$$

$$\ddot{q}_a = e^{-q_a} - e^{-q_{a+1}} - (e^{-q_{a-1}} - e^{-q_a})$$

$$= -e^{-q_{a-1}} + 2e^{-q_a} - e^{-q_{a+1}}$$

$$\Leftrightarrow \quad \boxed{\ddot{q}_a = \sum_{b=1}^{N-1} K_{ab} e^{-q_b}} \qquad a = 1, \dots, N-1$$

where K_{ab} is the Cartan-matrix for $SU(N)$

- this can be generalised to other Lie-algebras, using the Cartan classification of semi-simple LA: $[H^i, E^\alpha] = \alpha^i E^\alpha$, $K_{ij} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_j^2}$

into $H_{i=1,\dots,r}$, r rank, commuting
E^α non-com. generators $\Big\}$ span the Lie algebra