## An explicit realisation of the Toda-lattice: random matrix models

- consider $H = H^\dagger$ $N\times N$, hermitian matrix with $H_{ij} \in \mathbb{C}$ (indep.)
- thinking of $H$ to be a Hamiltonian with random elements

we have $$Z_N = \int dH \; e^{-\mathrm{Tr}\, V(H)}$$ partition function

int. over all indep. matrix elements $H_{ij}$

and for $V(x) = \sum_{k=0}^{\infty} t_k x^k$ potential in the Boltzmann weight

e.g. $V_2(x) = t_2 x^2$ Gauss $\Rightarrow e^{-\mathrm{Tr} V_2(H)}$ factorises on $H$

- $H$ is diagonalisable $H = U \Lambda U^\dagger$, $U \in U(N)$ unitary

$\Lambda = \mathrm{diag}(\lambda_1, .., \lambda_N)$

- one can show that in terms of $\lambda_i$

$$\boxed{Z_N = c_N \prod_{j=1}^{N} \int_{-\infty}^{\infty} d\lambda_j \, e^{-V(\lambda_j)} \; \Delta_N(\{\lambda\})^2}$$

const ($c_N$)

where $\Delta_N(\{\lambda\}) = \prod_{i>j}^{N} (\lambda_i - \lambda_j) \overset{\text{exercise}}{=} \det_{1\le i,j\le N}\left(\lambda_i^{j-1}\right) = \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{N-1} \\ \vdots & & & & \\ 1 & \lambda_N & \lambda_N^2 & \cdots & \lambda_N^{N-1} \end{vmatrix}$

Vandermonde det.

lin. algebra

$= \begin{vmatrix} \tilde{P}_0(\lambda_1) & \tilde{P}_1(\lambda_1) & \cdots & \tilde{P}_{N-1}(\lambda_1) \\ \tilde{P}_0(\lambda_N) & \cdots & & \end{vmatrix}$

with $\tilde{P}_k(\lambda) = \lambda^k + \ldots$ arb. polynomials

- random matrix models have many applications in Physics (condensed matter, quantum gravity, etc.) and Mathematics (number theory, combinatorics, etc.)
- the correlations of the eigenvalues $\lambda_i$ (and their repulsion) is universal for $N \to \infty$, that is indep. of the precise form of $V(x)$

- r MM are integrable for finite-$N$, in particular

$\frac{Z_{j+1}}{Z_j}$ satisfies the Toda-lattice eqs.

[ • eigenvalues interact as a Coulomb gas: put all into exp:

$$Z_N = c_N \int \prod d\lambda_i \, e^{S}, \quad S = \underbrace{-\sum_{j=1}^{N} V(\lambda_j)}_{\text{confining pot.}} + \underbrace{\sum_{i \neq j} \log|\lambda_i - \lambda_j|}_{\text{Coulomb interact. in 2D: repulsion}}$$ ]

- <u>method of orthogonal polynomials</u>

for given weight $e^{-V(x)}$ construct <u>orthogonal polynomials</u> (via Gram-Schmidt)

$$\int_{-\infty}^{\infty} d\lambda \, e^{-V(\lambda)} P_i(\lambda) P_j(\lambda) = \delta_{ij} h_j$$

$h_j > 0$ squared norms of $P_j$ ($\|P_j\|^2$)

- <u>example</u>: $V_2(x) = t_2 x^2 \Rightarrow P_j(x) \sim H_j(x)$ Hermite polynomials, $h_j$ known

properties:

- the $P_j(x)$ form a complete set of functions

$$\lambda P_j(\lambda) = \sum_{e=0}^{j+1} \alpha_j(i) P_i(\lambda)$$

ex.

$$\Rightarrow \boxed{\lambda_j P_j(\lambda) = P_{j+1}(\lambda) + S_j P_j(\lambda) + r_j P_{j-1}(\lambda)}$$

3-step recurrence relation (true for all OP on $\mathbb{R}$)

where $r_j = \frac{h_j}{h_{j-1}}$ $\quad j = 1, 2, \ldots$

- all correlation functions of eigenvalues can be computed in terms of $P_j(x)$, when choosing $\tilde{P}_j = P_j$ OP in the Vandermonde
- it holds

$$Z_N = c_N N! \prod_{j=0}^{N} h_j \qquad \text{(using that } P_j \text{ are OP)}$$

Define: $\underline{h_i \equiv e^{\phi_i}}$ $\left( \sim \frac{Z_{i+1}}{Z_i} > 0 \right)$

$\Rightarrow$ $\phi_i$ satisfies the Toda-Lattice eq

$$\boxed{\partial^2_{t_1} \phi_i = e^{\phi_{i+1} - \phi_i} - e^{\phi_i - \phi_{i-1}}} \qquad (\phi_i = -\varphi_i)$$

We need 2 Lemmas to show that

1) $\underline{s_i = -\partial_t \phi_i}$, $\partial_1 = \frac{\partial}{\partial t_1}$

2) $\underline{\partial_1 P_i(x) = r_i P_{i-1}(x)}$ (no summation conv.)

to 1) $\partial_1 h_i = \partial_1 e^{\phi_i} = \partial_1 \phi_i e^{\phi_i} = \underline{\partial_1 \phi_i h_i}$

$$= \partial_1 \int d\lambda e^{-V(\lambda)} P_i^2(\lambda) \qquad , \partial_1 V(\lambda) = \lambda$$

$$= \int d\lambda e^{-V(\lambda)} \Big( -\lambda P_i^2(\lambda) + \cancel{2 P_i(\lambda) \underbrace{\partial_1 P_i(\lambda)}_{\partial_1(\lambda^i + ..) = \text{polyn of order } i-1}} \Big)$$

3 step rec.

$$= -\int d\lambda e^{-V(\lambda)} P_i(\lambda) \left( \cancel{P_{i+1}(\lambda)} + s_i P_i(\lambda) + \cancel{r_i P_{i-1}(\lambda)} \right)$$

$$= \underline{-s_i h_i} \quad \checkmark$$

to 2) $\partial_\lambda P_i(\lambda) = \partial_\lambda(\lambda^i + \mathcal{O}(\lambda^{i-1})) = \mathcal{O}(\lambda^{i-1})$

ansatz: $\partial_\lambda P_i(\lambda) = \sum_{l=0}^{i-1} \beta_l P_l(\lambda)$, $\beta_l = \frac{1}{h_l} \int d\lambda\, e^{-V(\lambda)} P_l(\lambda) \partial_\lambda P_i(\lambda)$

use $0 = \partial_\lambda \underbrace{\int d\lambda\, e^{-V(\lambda)} P_i(\lambda) P_l(\lambda)}_{\sim \delta_{il}} = \int d\lambda\, e^{-V}(-\lambda P_i P_l + (\partial_\lambda P_i) P_l + P_i \partial_\lambda P_l)$

$=0$ for $l = 0,1,\dots,i-1$ ; 3 step rec ; $\sim \beta_l$ ; $\mathcal{O}(\lambda^{l-1})$

$\Leftrightarrow 0 = -h_i \delta_{i,l+1} + h_l \beta_l + 0$

$\Leftrightarrow \beta_l = \begin{cases} \frac{h_{l+1}}{h_l} = r_{l+1} & l = i-1 \\ 0 & \text{else} \end{cases} \Rightarrow \partial_\lambda P_i(\lambda) = r_i P_{i-1}(\lambda)$ ✓

- Toda-lattice eq.:

$$\partial_\lambda^2 e^{\phi_i} = \partial_\lambda(\partial_\lambda \phi_i e^{\phi_i}) = (\partial_\lambda^2 \phi_i) e^{\phi_i} + (\partial_\lambda \phi_i)^2 e^{\phi_i}$$

$$= \partial_\lambda^2 \int d\lambda\, e^{-V} P_i^2(\lambda) = \partial_\lambda \int d\lambda\, e^{-V}(-\lambda P_i^2(\lambda) + \cancel{2 P_i \partial_\lambda P_i})$$

$$= \int d\lambda\, e^{-V}(\lambda^2 P_i^2 - 2\lambda P_i \partial_\lambda P_i)$$

3 step and 2) rec $= \int d\lambda\, e^{-V}\left((P_{i+1}(\lambda) + s_i P_i(\lambda) + r_i P_{i-1}(\lambda))^2 - 2(P_{i+1} + s_i P_i + r_i P_{i-1}) r_i P_{i-1}\right)$

$$= h_{i+1} + s_i^2 h_i + \cancel{r_i^2 h_{i-1}} - \cancel{2} r_i^2 h_{i-1}$$

$$\overset{1)}{=} h_{i+1} + (\partial_\lambda \phi_i)^2 h_i - r_i^2 h_{i-1}$$

$$\Leftrightarrow \partial_\lambda^2 \phi_i = \frac{h_{i+1}}{h_i} - \frac{h_i^{\cancel{2}}}{h_{i-1}^{\cancel{2}}} \frac{\cancel{h_{i-1}}}{\cancel{h_i}} = e^{\phi_{i+1} - \phi_i} - e^{\phi_i - \phi_{i-1}} \checkmark$$

Note: $\mathcal{L}_{q \geq -1} Z_N = 0$ with $\mathcal{L}_q = \sum_{k=0}^{\infty} k t_k \partial_{t_{k+q}} + \sum_{k=0}^{\infty} \partial_{t_k} \partial_{t_{q-k}}$

with $\mathcal{L}_q$ satisfying the Virasoro algebra

# Integrable Quantum Systems – Quasi Exactly Solvability

- the purpose of this chapter on QES is two fold:
  - we will from now on consider QM system rather than classical ones

$$H\Psi \equiv \left(-\Delta + V(\vec{x})\right)\Psi(\vec{x}) = E\,\Psi(\vec{x})$$

and seek solutions of the Schrödinger eq.

  - as a first step toward quantum integrable (or exactly solvable) syst. we will study QES where only finitely many energies $E_n$ and corresp. wave functions $\Psi_n$ are known.

- an illustration
  - the Hamiltonian $H$ can be rep. by an infinite dim. matrix with elements $H_{nm} = \langle \phi_n | H | \phi_m \rangle$ where $\phi_n$ are orth. functions that form a basis of the Hilbert space.
  - the diagonalisation of $H$ by changing basis $\phi_n \to \Psi_n$ is highly non-trivial (in contrast to $\dim H < \infty$). There are 3 cases

**exactly solvable**

$$H = \begin{pmatrix} H_{00} & & & 0 \\ & H_{11} & & \\ & & H_{22} & \\ 0 & & & \ddots \end{pmatrix}$$

when the diagonal. is possible, e.g. for the Harmonic Oscillator

**quasi-exactly solvable**

$$H = \left(\begin{array}{cccc|c} H_{00} & H_{01} & \cdots & H_{0M} & \\ H_{10} & H_{11} & \cdots & & 0 \\ \vdots & & & & \\ H_{M0} & \cdots & & H_{MM} & \\ \hline & & 0 & & /// \end{array}\right)$$

this is the QES case, $H$ has a block structure and the first $M$ $E_n, \Psi_n$ can be found by exact diag ($E_{n>M}$ remains unknown)

**exactly non-solvable**

$$H = \begin{pmatrix} H_{00} & \cdots \\ \vdots & \\ \vdots & \end{pmatrix}$$

no diag of $H$ or sub blocks is known, this is the case for generic $V(\vec{x})$)

- the set of exactly solvable $V$'s is small

$\to$ hope to find many more $V$'s that are QES

- example $M = 1$:

  - take any real valued function $\xi(\vec{x}) \neq 0$ that has a definite sign $\geq 0$ and is square integrable:

$$\int_{\mathbb{R}} \xi^2(\vec{x})\, d^D x = 1$$

$\Rightarrow$ $\xi(\vec{x})$ is the ground state solution (no nodes) to the Hamiltonian $H$ in $D$ dimensions with potential $V(\vec{x}) = \frac{\Delta \xi(\vec{x})}{\xi(\vec{x})}$, with energy $E$.

- obviously there is a large set of such functions, but:
  - the resulting $V$ may not be very relevant physically
  - we would like to know at least a few excited states ($M > 1$), there is no systematic starting this way

- the Harmonic Oscillator revisited ($D = 1$)

$$\boxed{H = -\frac{\partial^2}{\partial x^2} + \alpha x^2}, \quad \alpha > 0$$

standard solution $a^+ = \sqrt{\alpha}\, x - \partial_x$, $a^- = \sqrt{\alpha}\, x + \partial_x$

$$\Rightarrow \underline{H = a^+ a^- - \sqrt{\alpha}}, \quad \underline{[a^+, a^-] = -2\sqrt{\alpha}}$$

- the ground state $|0\rangle = e^{-\frac{\sqrt{\alpha}}{2} x^2}$ is annihilated by $a^-$: $a^- |0\rangle = 0$

- the excited states are $|n\rangle = (a^+)^n |0\rangle$

- it is useful to slightly change basis for $a^-$ lowering and $a^+$ raising operator, as this choice can be generalised to other $V$'s:

$$\boxed{H = \sqrt{\alpha}\left(2a^+a^- + a^0\right) - (a^-)^2}$$

ex: check this

where $a^+ = x, \quad a^0 = 1, \quad a^- = \sqrt{\alpha}\, x + \partial_x$

→ these operators still satisfy the Heisenberg algebra:

$$[a^-, a^+] = a^0 \quad , \quad [a^+, a^0] = 0 = [a^-, a^0]$$

- we still have $a^-|0\rangle = 0, \ |n\rangle = (a^+)^n|0\rangle$ and $a^0|0\rangle = |0\rangle$

as well as $a^+|n\rangle = |n+1\rangle, \quad a^0|n\rangle = |n\rangle, \ a^-|n\rangle = n|n-1\rangle$

$$\Rightarrow \quad \boxed{H|n\rangle = \sqrt{\alpha}\,(2n+1)|n\rangle - n(n-1)|n-2\rangle} \qquad (*)$$

- two important consequences follow:
  - consider $\Phi_n = \text{span}\{|0\rangle, |1\rangle, \ldots, |n\rangle\}$ the $n+1$ dim subspace of the Hilbert space $\Phi = \{|n\rangle \mid n \in \mathbb{N}\}$

    $\Rightarrow \quad H\Phi_n \subseteq \Phi_n$ the action of $H$ remains in $\Phi_n$

    $\Rightarrow$ the $\infty$-dim of diagonalising $H$ on $\Phi$ is equivalent to the infinite set $M \in \mathbb{N}$ of finite-dim problems of diagonalising $H\varphi = E\varphi$ for $\varphi \in \Phi_M$.

    This is the essence of the exact solvability of the HO

  - the eigenstates of $H$ have parity

    $\Rightarrow$ consider the 2 subsets $\Phi_M^p = \text{span}\{|p\rangle, |2+p\rangle, \ldots, |2M+p\rangle\}$

    for $p = 0, 1$

- making a general ansatz for $\varphi \in \Phi_M$:

$$\varphi = \sum_{j=0}^{M} \eta_{M-j} |2j+p\rangle$$ we find that using ⊛

in $H\varphi = E\varphi$ we have $E = \sqrt{\alpha'}(4M+2p+1)$

together with a recursion for the coefs $\eta_m$:

$$-4\sqrt{\alpha'}\,\eta_{m+1} = \left(2(M-m)+p\right)\left(2(M-m)+p+1\right)\eta_m \qquad 4$$

that can be solved. Together with

$|0\rangle = e^{-\frac{\sqrt{\alpha'}x^2}{2}}$, $|n\rangle = x^n |0\rangle$ we find $\varphi = P_M(x)|0\rangle$ with

known $P_M(x)$ (ex definierte $P_M(x)$).

This completes the diagonalisation and thus the solution of the HO.

<u>Construction of exactly solvable systems using the Heisenberg algebra</u>

- the simplest realisation of the Heisenberg algebra:

$$a^0 = 1, \quad a^+ = t, \quad a^- = \frac{\partial}{\partial t}$$

- change variables $t = A(x)$ for some differentiable funct. $A(x)$

$$\Rightarrow \boxed{a^0 = 1, \quad a^+ = A(x), \quad a^- = \frac{1}{A'(x)}\left(\frac{\partial}{\partial x} - B(x)\right)}, \quad A'(x) = \frac{\partial}{\partial x}A(x) \neq 0$$

we may add here a second general funct. $C(x)$ as it commutes with $a^0$ and $a^+$ (parametrise $C(x) = -\frac{B(x)}{A'(x)}$)

How do we get the most general Hamiltonian $H$ based on these $a^0, a^+, a^-$ (in dim $D=1$)? Requirements:

- $H$ is at most quadratic in $\partial_x$ $\Rightarrow$ only powers $a^-, (a^-)^2$
- we need the property $H\,\underline{\Phi}_M \subseteq \underline{\Phi}_M$ to reduce the diag. of the infinite-dim $H$ to finite dim. subspaces, where

$\underline{\Phi}_M = \text{Span}\{|0\rangle, \ldots, |M\rangle = (a^+)^M|0\rangle\}$

$\Rightarrow$ because of the Heisenberg alg. this only works for combinations of operators $(a^+)^n (a^-)^m$ with $n \leq m$

(this is the most general comb.)

With $m \leq 2$ from above we get as ansatz for $H$:

$$\underline{H = A_1 (a^+)^2 (a^-)^2 + A_2 a^+ (a^-)^2 + A_3 (a^-)^2 + A_4 a^+ a^- + A_5 a^- + A_6}$$

with constants $A_{1,\ldots,6}$ $\quad (\Rightarrow H\,\underline{\Phi}_M \subseteq \underline{\Phi}_M \;\; \forall M)$

- multiplying out we obtain

$$\boxed{H = P(x)\frac{\partial^2}{\partial x^2} + Q(x)\frac{\partial}{\partial x} + R(x)}$$

with $P, Q, R$ functions of $A(x), B(x)$ (ex: determine these)

$\rightarrow$ for $H$ to be of the form of a Schrödinger op. we need

$$\boxed{P(x) = -1, \quad Q(x) = 0}$$

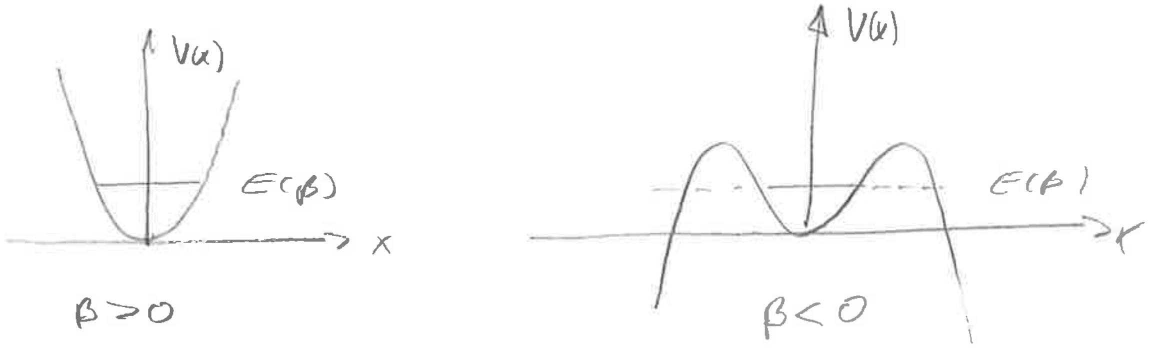
2 eqs. for 2 unknown functions $A(x), B(x)$

* famouse exactly solvable $V(x)$ fall into this class:

Morse potential, Pöschel-Teller-pot, HO + centrifugal barrier (ex: show this)

## An exactly non-solvable case: the quartic oscillator

$$H = -\frac{\partial^2}{\partial x^2} + \alpha x^2 + \beta x^4 \qquad \alpha > 0$$




- QM $\beta < 0$ is unstable as tunneling may occur
- We expect that for $\beta$ small we can treat $\beta x^4$ as a perturbation to the HO and do perturbation theory:

weak coupling $\frac{\beta}{\alpha^{3/2}} \ll 1$ and $\frac{\beta}{\alpha^{3/2}} \gtrsim 1$ strong coupling

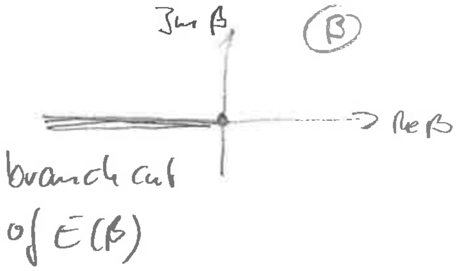
Can we trust the perturbative expansion?

Dyson's instability argument:

for any $\beta$ even infinitesimally small $\exists\, x_c$ s.th. the perturbation

$\beta x^4 > \alpha x^2$ for all $x > x_c$

- together with the instability by changing the sign of $\beta$ this implies $\beta = 0$ is a singularity of $E(\beta)$ for $\beta \in \mathbb{C}$




$\Rightarrow$ $E(\beta)$ has zero radius of convergence at $\beta = 0$ (can only do "asymptot. series")

- in fact the complex structure of $E(\beta)$ is much more complicated [Bender, Wu 1969]