The sextic oscillator - an example for QES

- Consider the anharmonic oscillator with quartic and sextic term:

$$\boxed{H = -\frac{\partial^2}{\partial x^2} + \alpha x^2 + \beta x^4 + \gamma x^6} \quad , \gamma > 0$$

where we need γ positive for the potential to be confining. Surprisingly:

* in this model $[\frac{n}{2}]+1$ exact solutions can be found

given that $$\underline{\frac{1}{\sqrt{\gamma}}\left(\frac{\beta^2}{4\gamma} - \alpha\right) = 2n+3}$$

so adding a higher order term makes the quartic anharmonic oscillator "easier"!

Heisenberg algebra representation:

$$\underline{a^+ = x \ , \ a^0 = 1 \ , \ a^- = \sqrt{\gamma}\, x^3 + \frac{\beta}{2\sqrt{\gamma}}\, x + \frac{\partial}{\partial x}} \qquad \text{Ansatz}$$

following our previous construction of exactly solvable H's

we put all structure into a^- (into Bcos) keeping $A(x) = x$.

This keeps the construction of excited wave functions simple.

It can be shown that (exercise)

$$\boxed{\begin{aligned} H = {} & \frac{\beta}{2\sqrt{\gamma}}\left(2a^+a^- + a^0\right) - \left(a^-\right)^2 \qquad \leftarrow \text{HO part} \\ & + \left(a^+\right)^2\left\{ 2\sqrt{\gamma}\, a^+a^- + \left(\alpha - \frac{\beta^2}{4\gamma} + 3\sqrt{\gamma}\right) a^0 \right\} \qquad \leftarrow \text{part} \end{aligned}}$$

that violates $(a^+)^m (a^-)^n$ with $m \le n \le 2$, so apriori it does not lead to a (partly) diagonalisable H basis states:

we again define $a^-|0\rangle = 0$, $a^0|0\rangle = 1$, $(a^+)^n|0\rangle = |n\rangle$

$\Rightarrow$ $a^+|n\rangle = |n+1\rangle$, $a^0|n\rangle = |n\rangle$, $a^-|n\rangle = n|n-1\rangle$ as before

Q: what is the coordinate representation of the ground state?

ansatz $$|0\rangle = \exp\left\{-\frac{ax^4}{4} - \frac{bx^2}{2}\right\}$$

$$\Rightarrow a^-|0\rangle = \left(\sqrt{\gamma}x^3 + \frac{\beta}{2\sqrt{\gamma}}x + \frac{\partial}{\partial x}\right)|0\rangle = \left((\sqrt{\gamma}-a)x^3 + \left(\frac{\beta}{2\sqrt{\gamma}} - b\right)x\right)|0\rangle$$
$$\overset{!}{=} 0 \quad \Leftrightarrow \quad \boxed{a = \sqrt{\gamma},\ b = \frac{\beta}{2\sqrt{\gamma}}}$$

- $$H|n\rangle = \frac{\beta}{2\sqrt{\gamma}}(2n+1)|n\rangle - n(n-1)|n-2\rangle + (a^+)^2\left\{2\sqrt{\gamma}n + \left(\alpha - \frac{\beta^2}{4\gamma} + 3\sqrt{\gamma}\right)\right\}|n\rangle$$
$$= \frac{\beta}{2\sqrt{\gamma}}(2n+1)|n\rangle - n(n-1)|n-2\rangle + \left\{(3+2n)\sqrt{\gamma} + \alpha - \frac{\beta^2}{4\gamma}\right\}|n+2\rangle$$

$\Rightarrow$ $H\,\bar{\Phi}_n \subseteq \bar{\Phi}_{n+2}$ does not remain in the subspace!

- just as the HO our H has parity, so H acts on subspaces $\bar{\Phi}^p_m = \mathrm{span}\{|p\rangle, |p+2\rangle, \dots |p+2m\rangle\}$ $p = 0, 1$:

$$H\,\bar{\Phi}^p_m \subseteq \bar{\Phi}^p_{m+1}$$

$\Rightarrow$ for either $p=0$ or $p=1$ we can create an invariant subspace by requiring

$$\boxed{(3+2n_0)\sqrt{\gamma} + \alpha - \frac{\beta^2}{4\gamma} = 0} \qquad n_0 = 2m_0 + p$$

for a given set of couplings α, β, γ

$\Rightarrow$ • for $n_0 = 2m_0$ even we can diag H on span$\{|0\rangle, |2\rangle, \ldots, |2m_0\rangle\}$

so we know m_0+1 energies from the ground state $|0\rangle$ to $|2m_0\rangle$ with even parity

• for $n_0 = 2m_0+1$ odd we can diag H on span $\{|1\rangle, \ldots, |2m_0+1\rangle\}$

$\Rightarrow$ we know m_0+1 energies from the 1st excited state to $|2m_0+1\rangle$ with odd parity $p=1$

* for thes QES potentials it is useful to use variables

$$0 < a = \sqrt{\gamma}, \quad b = \frac{\beta}{2\sqrt{\gamma}} \Rightarrow \alpha = b^2 - a(3 + 4m_0 + 2p) \text{ instead:}$$

$$\Leftrightarrow \beta = 2ab$$

$$\boxed{H = -\frac{\partial^2}{\partial x^2} + \underbrace{[b^2 - a(3+4m_0+2p)]x^2 + 2abx^4 + a^2x^6}_{V(x)}} \quad , a>0 \quad m_0 \in \mathbb{N} \quad p = 0,1$$

• the basis we have to diagonalise is

$$|n\rangle = |2m+p\rangle = (a^+)^n|0\rangle = x^{2m} \cdot x^p e^{-\frac{ax^4}{4} - \frac{bx^2}{2}}$$

$\Rightarrow$ the eigenfunctions of H for given p will be of the form

$$\psi(x) = x^p P_m(x^2) e^{-\frac{ax^4}{4} - \frac{bx^2}{2}}$$

with $P_m(y)$ a polynomial of degree m in $y = x^2$ ($\Rightarrow P_m$ even)

• we can determine the m_0+1-p eigenfunctions for given n_0 and p and their respective polynomial as follows:

$$H\, x^p P^{(l)}_{m_0}(x^2)\, e^{-\frac{ax^4}{4}-\frac{bx^2}{2}} = E\, x^p P^{(l)}_{m_0}(x^2)\, e^{-\frac{ax^4}{4}-\frac{bx^2}{2}} \quad , \; l = 1, \dots, m_0+1-p$$

leading to a differential eq. for the polynomial with m_0+1-p solutions

exercise:

$$\boxed{\left\{ -\frac{\partial^2}{\partial x^2} - \left(\frac{2p}{x} - 2ax^3 - 2bx \right) \frac{\partial}{\partial x} + b(2p+1) - 4am_0 x^2 - E \right\} P^{(l)}_{m_0}(x^2) = 0}$$

examples:

- $m_0 = 0$: $\Rightarrow$ we know groundstate for $p=0$ or $p=1$ 1st excited state

treating $p = 0,1$ simultaneously we have $P_{m_0=0} = \text{const} \equiv 1$

$\overset{\text{above}}{\Longrightarrow}$ diff eq. $\boxed{E_p = b(2p+1)} = \begin{cases} b & p=0 \\ 3b & p=1 \end{cases}$

with wave fnct. $\boxed{\psi_p(x) = \begin{Bmatrix} 1 \\ x \end{Bmatrix} \cdot e^{-\frac{ax^4}{4}-\frac{bx^2}{2}}}$ for $p = \begin{cases} 0 \\ 1 \end{cases}$ $\Rightarrow$ 1 node = 1st state

(of course the wave fnct has to be properly normalised to give $\int \psi^2 = 1$

- $m_0 = 1$: $\Rightarrow$ we have $P_{m_0=1}(x^2) = x^2 + q$ for $p = 0,1$

$$\Rightarrow -2 - 2(2p - 2ax^4 - 2bx^2) + \left(b(2p+1) - 4ax^2 - E \right)(x^2+q) = 0$$

$\mathcal{O}(x^4)$: ✓ $\quad \mathcal{O}(x^2)$: $\quad q = \dfrac{(2p+5)b - E}{4a}$

$\mathcal{O}(x^0)$: $\quad q\left((2p+1)b - E\right) = 2(2p+1)$

$\Rightarrow$ quadratic eq. for E: $0 = \left(b(2p+1) - E\right)\left((2p+5)b - E\right) - 8a(2p+1)$

2 solutions (ground + 2nd or 1st + 3rd)

$$\boxed{\begin{aligned} E_\pm &= (2p+3)b \pm 2\sqrt{b^2 + 2a(2p+1)} \\ \psi_\pm &= (x^2+q_\pm)\, x^p e^{-\frac{ax^4}{4}-\frac{bx^2}{2}} , \quad q_\pm = \frac{1}{4a}\left(2b \mp 2\sqrt{b^2+2a(2p+1)}\right) \end{aligned}}$$

- for generic m_0 we expect an algebraic eq. of order

$\rightarrow$ parametrise $P_{m_0}(x^2)$ in terms of it's zeros as an ansatz:

$$\psi(x) = \underbrace{\prod_{i=1}^{m_0} \left(\frac{x^2}{2} - \xi_i\right)}_{P_{m_0}(x^2)} x^p e^{-a\frac{x^4}{4} - b\frac{x^2}{2}}$$

inserted in the Schrödinger eq acting on P_{m_0}

ex

$$\Rightarrow \sum_{i=1}^{m_0} \frac{1}{\frac{x^2}{2} - \xi_i} \underbrace{\left\{ \sum_{\substack{k=1 \\ k \neq i}}^{m_0} \frac{4\xi_i}{\xi_i - \xi_k} + 2p + 1 - 4b\xi_i - 8a\xi_i^2 \right\}}_{\equiv \{\,\}_i}$$

$$+ E - (4m_0 + 2p + 1)b - 8a\sum_{i=1}^{m_0} \xi_i = 0$$

this can only be satisfied if the coeff. of the poles vanish individually

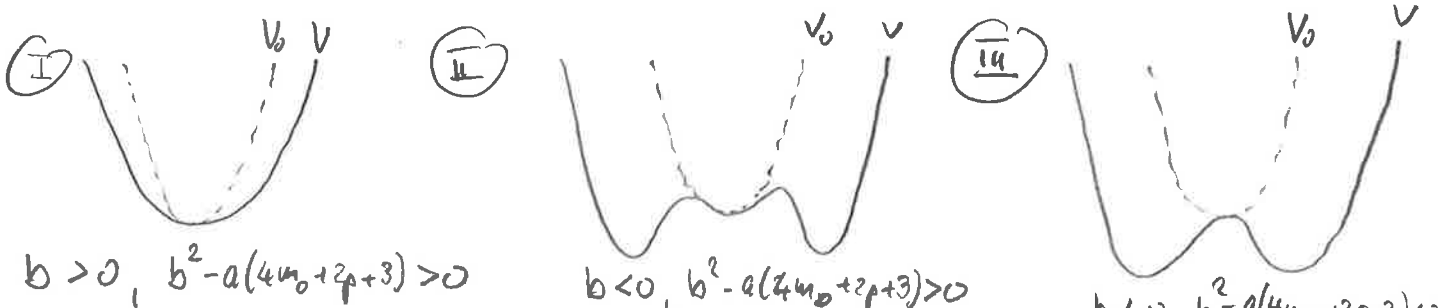
$$\Rightarrow \boxed{E = (4m_0 + 2p + 1)b + 8a\sum_{i=1}^{m_0} \xi_i}$$ energy

and the algebraic eq. $\{\,\}_i = 0$ $i = 1, \dots, m_0$ will have m_0 sets of solutions $\{\xi_i^{(\ell)}\}_{i=1\dots m_0}$ with $\ell = 1, \dots, m_0 \Rightarrow$ engies & ψ's

- a note on perturbation theory:

for $V(x)$ on p. 139 we have 3 situations:

$$V(x) = b^2x^2 + a\left[2bx^4 - (4m_0+2p+3)x^2\right] + a^2x^6$$ vs $V_0(x) = b^2x^2$ ← HO



$b > 0,\ b^2 - a(4m_0+2p+3) > 0$ $\qquad$ $b < 0,\ b^2 - a(4m_0+2p+3) > 0$ $\qquad$ $b < 0,\ b^2 - a(4m_0+2p+3) < 0$

$\rightarrow$ only for I $b > 0$ we can expect that $\psi_0(x) = e^{-b\frac{x^2}{2}}$ and

$\psi(x) \sim e^{-a\frac{x^4}{4} - \frac{bx^2}{2}}$ will good agreement for small perturbations in $a \ll 1$

- the example of the QES sextic oscillator can be generalised along the following lines:
  - relax the condition $(a^+)^l (a^-)^n$ $l \le n \le 2$

    to $l \le n + k$

    $\rightarrow$ it can be shown that then $H \Phi_m \subseteq \Phi_{m+k}$

  (of course we still need to assume that H is at most a quadratic diff. op $H = -\frac{\partial^2}{\partial x} + V$)

- in order to restrict the action of H to a finite subspace

  $$H |n\rangle = \ldots + A_1 |n+1\rangle + A_2 |n+2\rangle + \ldots A_k |n+k\rangle$$

  we need to impose more constraints on the couplings of V by requiring $A_1 = A_2 = \ldots = A_k = 0$ $\Rightarrow H \Phi_n \subseteq \Phi_n$

- H will then be of the form

  $$H = u_0 + u_1 (a^+ a^- - \mu) + u_2 (a^+ a^- - \mu + 1)(a^+ a^- - \mu)$$

  with the u_i's being at most quadratic in a^+ and a^-

For more details see the book

"Quasi-exactly solvable models in quantum mechanics" by A. G. Ushveridze, Institute of Physics Publishing, Bristol, 1994 (ISBN 0750302666)