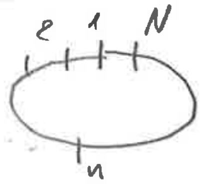
Quantum Integrability – discrete case : spin chains

- in this chapter we will generalise the concept of inverse scattering and Lax pair to quantum systems [Lit: L.D. Faddeev arXiv:hep-th/9605187]

discrete space : lattice, 1D: chain, with pbc



spacing Δ, coord $x = n\Delta$

continuum limit $N \to \infty$, $\Delta \to 0$

$\to$ replace $\delta(x-y) \to \frac{\delta_{nm}}{\Delta}$

- in QM observables are represented by operators (and their algebra), here at each point X_n^α. Previous continuum commutation relations now depend on the position n. A set of operators is called ultralocal if $[X_n^\alpha, Y_m^\beta] \sim \delta_{nm}$

- examples: canonical variables $\varphi_n^\alpha, \pi_n^\alpha$, $\alpha = 1, \dots, \ell$ dof per lattice site

$$[\varphi_n^\alpha, \varphi_m^\beta] = 0 = [\pi_n^\alpha, \pi_m^\beta]$$

$$[\varphi_m^\alpha, \pi_n^\beta] = i\hbar \underbrace{I}_{\text{identity}} \delta_{nm} \delta_{\alpha\beta}$$

, e.g. φ and π acting as multipl. and diff.

spin variables

$$[S_m^\alpha, S_n^\beta] = i\hbar\, \epsilon_{\alpha\beta\gamma} S_n^\gamma \delta_{mn}$$

, $\epsilon_{123} = 1$, tot. antisym.

representations are labelled by $s = 0, \frac{1}{2}, 1 \dots$

spin $\frac{1}{2}$ $S_n^\alpha = \frac{\hbar}{2} \sigma^\alpha$, Pauli matrices $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

• Hilbert space: at each lattice point n we have a Hilbert subspace h_n

$\Rightarrow$ for an ultralocal set of observables (operators) we have for the total Hilbert space $\mathcal{H} = \prod_{n=1}^{N} \otimes h_n = h_1 \otimes h_2 \otimes \dots \otimes h_n$ direct prod

so X_n^α acts as $\mathrm{Id} \otimes \dots \otimes \underset{n\text{-th}}{X^\alpha} \otimes \mathrm{Id} \dots \otimes \mathrm{Id}$

The $XXX_{\frac{1}{2}}$ quantum spin chain

$h_n = \mathbb{C}^2$ set of complex vectors, we have spin variables ~ Pauli S_n^α acting on each h_n:

Hamiltonian $\quad H = \sum_{\alpha, n} \left(S_n^\alpha S_{n+1}^\alpha - \frac{1}{4} \right)$

total spin $\quad S^\alpha = \sum_n S_n^\alpha$

. the S_n^α are periodic $S_{n+N}^\alpha = S_n^\alpha$

it holds $[H, S^\alpha] = 0$ (check)

Note the most general $XYZ_{\frac{1}{2}}$ chain would be $H = \sum_{\alpha, n} J^\alpha S_n^\alpha S_{n+1}^\alpha$ with different couplings for each α

Define Lax operator ("analogue" to classical L)

$$\boxed{L_{n,a}(\lambda) = \lambda I_n \otimes I_a + i \sum_\alpha S_n^\alpha \otimes \sigma^\alpha}$$

, $\lambda \in \mathbb{C}$ spectral param.

$L_{n,a}$ acts on $h_n \otimes V$, V auxiliary space, here also $\mathbb{C}^2$
I_n, I_a identity op. $S_n^\alpha, \sigma^\alpha$ spin on h_n and V

p.64

$$\sigma_\pm = \frac{1}{2}(\sigma^1 \pm i\sigma^2) = \begin{cases} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & "+" \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & "-" \end{cases}$$

matrix rep $\quad L_{n,a}(\lambda) \underset{ex}{=} \begin{pmatrix} \lambda + i S_n^3 & i S_n^- \\ i S_n^+ & \lambda - i S_n^3 \end{pmatrix} i$

Permutation operator $\quad P\, a\otimes b = b \otimes a \quad \Rightarrow P^2 = Id$

it is given by $\boxed{P_{n,a} = \frac{1}{2}\left(I_n \otimes I_a + \sum_\alpha \sigma_n^\alpha \otimes \sigma^\alpha \right)}$ ex check and write a matrix rep

$\Rightarrow$ we can write

$$L_{n,a}(\lambda) = (\lambda - \tfrac{i}{2}) \underbrace{I_n \otimes I_a}_{\equiv I_{n,a}} + i P_{n,a}$$

commutator of 2 Lax operators (only at equal n, ultralocal)

$L_{na_1} L_{na_2}$ act on $h_n \otimes V_1 \otimes V_2$

compare to $L_{na_2} L_{na_1}$:

claim: $\exists$ operator $R_{a_1 a_2}$ (acts on $V_1 \otimes V_2$) called R-matrix s.th.

$$(*) \quad \boxed{R_{a_1,a_2}(\lambda-\mu)\, L_{n,a_1}(\lambda)\, L_{n,a_2}(\mu) = L_{n,a_2}(\mu)\, L_{n,a_1}(\lambda)\, R_{a_1 a_2}(\lambda-\mu)}$$

where $R_{a_1,a_2}(\lambda) = \lambda I_{a_1,a_2} + i P_{a_1,a_2}$ acting on $V_1 \otimes V_2$

this is called fundamental commutation relation (FCR)

It can be verified using the following relations (ex.)

$$(**) \quad \boxed{\begin{gathered} P_{n,a_1} P_{n,a_2} = P_{a_1,a_2} P_{n,a_1} = P_{n,a_2} P_{a_2,a_1} \\ \text{and} \quad P_{a_2,a_1} = P_{a_1,a_2} \end{gathered}}$$

as well as the same relations with indices 1 and 2 interchanged.

- so far we have only considered the commutator of operators
- on states Ψ_n living on the lattice (chain) we have

Lax equation $\boxed{\Psi_{n+1} = L_n \Psi_n}$

$\Rightarrow L_{n,a}$ defines transport on the chain (geometrically: connection)

define $\underline{T^{n_2}_{n_1,a}(\lambda) = L_{n_2,a}(\lambda) L_{n_2-1,a}(\lambda) \ldots L_{n_1,a}(\lambda)}$

for $n_2 > n_1$. This ordered product defines $\Psi_{n_1} \to \Psi_{n_2+1}$

__full product__ $T_{N,a}(\lambda) = L_{N,a}(\lambda) \ldots L_{1,a}(\lambda)$

transports Ψ_1 back to itself, modulo a mismatch, called

__monodromy__, with matrix rep on V

$$T_{N,a} = \begin{pmatrix} A_N(\lambda) & B_N(\lambda) \\ C_N(\lambda) & D_N(\lambda) \end{pmatrix}$$

$T_{N,a}$ is a generating object for spin, Hamiltonian etc.
($\uparrow$ drop N in following)

__FCR for T__: $\boxed{R_{a_1 a_2}(\lambda-\mu) T_{a_1}(\lambda) T_{a_2}(\mu) = T_{a_2}(\mu) T_{a_1}(\lambda) R_{a_1 a_2}(\lambda-\mu)}$

It follows easily for any transport operator from $n \to n+2$ for any n (ex.)

denote $L_{n a_1}(\lambda) = L_1$; $L_{n,a_2}(\mu) = L_2$; $R_{a_1 a_2}(\lambda-\mu) = R_{12}$
$L_{n+1\, a_1}(\lambda) = L_1'$; $L_{n+1, a_2}(\mu) = L_2'$

due to ultralocality any L_j at site n commutes with L_j' at site $n+1$

$$\underbrace{}_{}T^{n+1}_{n a_2}(\mu) \quad T^{n+1}_{n a_1}(\lambda)$$

$$\Rightarrow \quad L_2' L_2 L_1' L_1 R_{12} = L_2' L_1' L_2 L_1 R_{12}$$

gives perm

$$= L_2' L_1' R_{12} L_1 L_2 \qquad = R_{12} L_1' L_2' L_1 L_2$$

gives perm

$$= R_{12} \underbrace{L_1' L_1}_{T^{n+1}_{n a_1}(\lambda)} \underbrace{L_2' L_2}_{T^{n+1}_{n a_2}(\mu)}$$

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properties of $T_{N,a}(\lambda)$:

$$L_{n,a}(\lambda) = \lambda I_{n,a} + i \sum_\alpha S_n^\alpha \otimes \sigma^\alpha \quad \Rightarrow T_{N,a}(\lambda) \text{ is a polynomial:}$$

$$\Rightarrow T_{N,a}(\lambda) = \prod_{n=1}^{N} L_{n,a}(\lambda) = \lambda^N I + i \lambda^{N-1} \sum_{n=1}^{N} \sum_\alpha S_n^\alpha \otimes \sigma^\alpha + \dots$$

$$= \lambda^N I + i \lambda^{N-1} \sum_\alpha \underset{\substack{\uparrow \\ \text{total spin}}}{S^\alpha} \otimes \sigma^\alpha + \dots$$

generating functional

consider $F_N(\lambda) = \mathrm{tr}_V T_{N,a}(\lambda) = \mathrm{tr}_V \begin{pmatrix} A_N(\lambda) & B_N(\lambda) \\ C_N(\lambda) & D_N(\lambda) \end{pmatrix} = \underset{\sim \lambda^N}{A_N(\lambda)} + \underset{\sim \lambda^N}{D_N(\lambda)}$

- because all σ^α are traceless

$$F_N(\lambda) = \mathrm{Tr}\, T_N(\lambda) = 2\lambda^N + \sum_{e=0}^{N-2} Q_e \lambda^e$$

- because $[F(\lambda), F(\mu)] = 0$ which follows from FCR

(ex: hint mult. FCR $\cdot R^{-1}$ and then take $\mathrm{Tr}_{V_1 \otimes V_2}$ on both sides)

$\Rightarrow$ <u>all operators Q_e are commuting</u> !

claim: H belongs to this family Q_c

proof: p. 145 : $\mathcal{L}_{n,a}(\lambda=\frac{i}{2}) = i P_{n,a}$; $\frac{d}{d\lambda}\mathcal{L}_{n,a}(\lambda) = I_{n,a}$ (1)

expand $F(\lambda)$ at $\lambda \approx \frac{i}{2}$: $T_{N,a}(\frac{i}{2}) = i^N P_{N,a} P_{N-1,a} \cdots P_{2,a} P_{1,a}$

$$\overset{ex}{=} i^N P_{12} P_{2,3} \cdots P_{N-1,N} P_{N,a}$$

using $tr_a P_{N,a} = \frac{1}{2} I_N \otimes tr I_a = I_N \Rightarrow$ $\boxed{U \equiv i^{-N} tr_a T_{N,a}(\tfrac{i}{2})} = P_{12} P_{23} \cdots P_{N-1,N}$

$$= i^{-N} F(\tfrac{i}{2})$$

- U is a shift operator on $\mathcal{H}$:

$P_{a_1 a_2}\, a \otimes b = b \otimes a \Leftrightarrow P_{n_1 n_2} X_{n_2} P_{n_1 n_2} = X_{n_1} \underbrace{P^2_{n_1 n_2}}_{= I_{n_1,n_2}} = X_{n_1}$

$\Leftrightarrow P_{n_1 n_2} X_{n_2} = X_{n_1} P_{n_1 n_2}$

$\Rightarrow X_n U = P_{12} \cdots X_n P_{n-1,n} P_{n,n+1} \cdots P_{N-1,N} = U X_{n-1}$

$P_{n-1,n} X_{n-1}$

- because of $(\sigma^\alpha)^+ = \sigma^\alpha \Rightarrow P^+_{n,a} = \frac{1}{2}(I_{n,a} + \sum_\alpha \sigma^\alpha_n \otimes \sigma^\alpha)^+ = P_{n,a}$

$\Rightarrow U^+ = P_{N,N-1} \cdots P_{32} P_{21}$, so $\boxed{UU^+ = U^+U = I}$ unitary

the shift can be written as

$$U^{-1} X_n U = X_{n-1}$$

this shift by 1 site can be interpreted as momentum

- $\frac{d}{d\lambda} T_a(\lambda)\Big|_{\lambda=\frac{i}{2}} \overset{(1)}{=} i^{N-1} \sum_n P_{Na} \cdots \underset{\uparrow\ \text{absent}}{\hat{P}_{n,a}} \cdots P_{1a}$, take tr_a and repeat above trick

$\Rightarrow \frac{d}{d\lambda} tr_a T_a(\lambda)\Big|_{\lambda=\frac{i}{2}} = \frac{d}{d\lambda} F(\lambda)\Big|_{\lambda=\frac{i}{2}} = i^{N-1} \sum_n P_{12} \cdots P_{n-1,n+1} \cdots P_{N-1,N} \quad \Big| \cdot U^{-1}$

$\Rightarrow \frac{d}{d\lambda} \ln F(\lambda)\Big|_{\lambda=\frac{i}{2}} = \frac{dF(\lambda)}{d\lambda} F^{-1}(\lambda)\Big|_{\lambda=\frac{i}{2}} = \frac{1}{i} \underbrace{\sum_n P_{n-1,n}}_{\text{p.b.c.} \Leftarrow \sum_n P_{n,n+1}} = i^{-1}\left(\frac{1}{2}\sum_n I_n \otimes I_{n+1} + \frac{1}{2}\sum_n \sigma^\alpha_n \sigma^\alpha_{n+1}\right)$

$$= i^{-1}(2H + 2N)$$

Bethe Ansatz equations

- we will show that the FCR for $T_a(\lambda)$ are equivalent to

$$\boxed{\begin{aligned} &[B(\lambda), B(\mu)] = 0 \\ &A(\lambda)\, B(\mu) = f(\lambda-\mu)\, B(\mu) A(\lambda) + g(\lambda-\mu)\, B(\lambda) A(\mu) \\ &D(\lambda)\, B(\mu) = f^*(\lambda-\mu)\, B(\mu) D(\lambda) - g(\lambda-\mu)\, B(\lambda) D(\mu) \end{aligned}}$$

where $f(\lambda) = \frac{\lambda - i}{\lambda}$, $g(\lambda) = \frac{i}{\lambda}$ FCR for matrix elements of T

- as we will show the operators A and B will generalise the HO creation and annihilation operators with
  $A \leftrightarrow a^-$, $B \leftrightarrow a^+a^- = \hat{n}$

FCR: $R_{a_1a_2}(\lambda-\mu) T_{a_1}(\lambda) T_{a_2}(\mu) = T_{a_2}(\mu) T_{a_1}(\lambda) R_{a_1a_2}(\lambda-\mu)$

all operators act on $(\mathcal{H} \otimes) V_1 \otimes V_2$. We will write them in 4×4 matrix rep. in the basis $e_+ = \begin{pmatrix}1\\0\end{pmatrix}$, $e_- = \begin{pmatrix}0\\1\end{pmatrix}$

recall $P_{a_1a_2} = \frac{1}{2} I_{a_1} \otimes I_{a_2} + \frac{1}{2}\sum_\alpha \sigma^\alpha_{a_1} \times \sigma^\alpha_{a_2}$:

$$= \begin{pmatrix} \frac{1}{2} + S^3_{a_1} & S^-_{a_1} \\ S^+_{a_1} & \frac{1}{2} - S^3_{a_1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

matrix rep in V_2 4x4 matrix rep in $V_1 \otimes V_2$

$$\Rightarrow R_{a_1a_2}(\lambda) = \lambda I_{a_1a_2} + i P_{a_1a_2} = \begin{pmatrix} \lambda+i & 0 & 0 & 0 \\ 0 & \lambda & i & 0 \\ 0 & i & \lambda & 0 \\ 0 & 0 & 0 & \lambda+i \end{pmatrix} \equiv \begin{pmatrix} a(\lambda) & & & \\ & b(\lambda) & c & \\ & c & b(\lambda) & \\ & & & a(\lambda) \end{pmatrix}$$

- $T_{a_1}(\lambda)$ acts on $V_1 \otimes V_2$ as $T_{a_1}(\lambda) \otimes \mathbb{1}_{a_2}$

  $T_{a_2}(\lambda)$ $\qquad\qquad\qquad\qquad \mathbb{1}_{a_1} \otimes T_{a_2}(\lambda)$

$$\Rightarrow T_{a_1}(\lambda) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \otimes \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} = \begin{pmatrix} A\mathbb{1} & B\mathbb{1} \\ C\mathbb{1} & D\mathbb{1} \end{pmatrix} = \begin{pmatrix} A(\lambda) & 0 & B(\lambda) & 0 \\ 0 & A(\lambda) & 0 & B(\lambda) \\ C(\lambda) & 0 & D(\lambda) & 0 \\ 0 & C(\lambda) & 0 & D(\lambda) \end{pmatrix}$$

$$T_2(\mu) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \otimes \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A(\mu) & B(\mu) & 0 & 0 \\ C(\mu) & D(\mu) & 0 & 0 \\ 0 & 0 & A(\mu) & B(\mu) \\ 0 & 0 & C(\mu) & D(\mu) \end{pmatrix}$$

$$\Rightarrow R_{a_1a_2}(\lambda-\mu)\, T_{a_1}(\lambda)\, T_{a_2}(\lambda) = R_{a_1a_2}(\lambda-\mu) \begin{pmatrix} A(\lambda)A(\mu) & A(\lambda)B(\mu) & B(\lambda)A(\mu) & B(\lambda)B(\mu) \\ A(\lambda)C(\mu) & A(\lambda)D(\mu) & B(\lambda)C(\mu) & B(\lambda)D(\mu) \\ C(\lambda)A(\mu) & C(\lambda)B(\mu) & D(\lambda)A(\mu) & D(\lambda)B(\mu) \\ C(\lambda)C(\mu) & C(\lambda)D(\mu) & D(\lambda)C(\mu) & D(\lambda)D(\mu) \end{pmatrix}$$

$$= \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \cdot \begin{pmatrix} & & \\ & \text{—''—} & \\ & & \end{pmatrix}$$

FCR

$$= T_{a_2}(\mu)\, T_{a_1}(\lambda)\, R_{a_1a_2}(\lambda-\mu) = \begin{pmatrix} \text{all matrix elements} \\ \text{with factors in} \\ \text{opposite order} \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$$

(check!)

- compare matrixelement wise

$(1,4) \Rightarrow [B(\lambda), B(\mu)] = 0$, $\quad (1,3)$: $a(\lambda-\mu) B(\lambda) A(\mu) = \ldots$ & exchange λ, μ

$\qquad (3,4) \quad \ldots = B(\mu) D(\lambda) a(\lambda-\mu)$

compare: $A(\lambda)B(\lambda) = \frac{a(\mu-\lambda)}{b(\mu-\lambda)} B(\mu)A(\lambda) - \frac{c}{b(\mu-\lambda)} B(\lambda)A(\mu)$

$$A(\lambda)B(\lambda) = \frac{\lambda-\mu-i}{\lambda-\mu} B(\mu)A(\lambda) + \frac{i}{(\lambda-\mu)} B(\lambda)A(\mu)$$

with HO $\boxed{a^- \underbrace{\hat{n}}_{a^+a^-} = (\hat{n}+1)a^-}$ as $[a^-, a^+] = 1$, $a^+a^- = \hat{n}$

likewise compare commutator of DB with

$$\boxed{a^+ \underbrace{\hat{n}}_{a^+a^-} = (\hat{n}-1)a^+}$$

- in further analogy we wish to find a ground state corresponding to $a^-|0\rangle = 0$: we will construct Ω s.th $C(\lambda)\Omega = 0$

- recall $L_{n,a}(\lambda) = \begin{pmatrix} \lambda+\frac{i}{2} & 0 & 0 & 0 \\ 0 & \lambda-\frac{i}{2} & i & 0 \\ 0 & i & \lambda-\frac{i}{2} & 0 \\ 0 & 0 & 0 & \lambda+\frac{i}{2} \end{pmatrix}$

- acting on $\omega_n \otimes I_a$ with $\omega_n = e_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ L becomes upper triangular:

$$L_{n,a}(\lambda)\,\omega_n = \begin{pmatrix} (\lambda+\frac{i}{2})\mathbb{1} & * \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \lambda-\frac{i}{2}\mathbb{1} \end{pmatrix}$$, where $*$ is some operator

$\Rightarrow$ define $\Omega = \prod_n \otimes \omega_n \;(\otimes I_a)$ with $\omega_n = e_+ \;\forall n$

$\Rightarrow T(\lambda)\Omega = \begin{pmatrix} (\lambda+\frac{i}{2})^N & * \\ 0 & (\lambda-\frac{i}{2})^N \end{pmatrix}$, def $\alpha(\lambda) = \lambda+\frac{i}{2}$, $\delta(\lambda) = \lambda-\frac{i}{2}$

from $T_a(\lambda)\,\Omega = \underbrace{\begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}}_{V \text{ matrix}} \Omega = \begin{pmatrix} A(\lambda)\Omega & B(\lambda)\Omega \\ C(\lambda)\Omega & D(\lambda)\Omega \end{pmatrix}$

we read off $\quad A(\lambda)\,\Omega = \alpha^N \Omega \,,\; D(\lambda)\,\Omega = \delta^N \Omega \,,\; C(\lambda)\Omega = 0$

$\Rightarrow$ Ω is also an eigenvector of $F(\lambda) = A(\lambda) + D(\lambda)$

- the "analogue" of $|n\rangle = (a^+)^n |0\rangle$ (which still has to be diag. to be an HO eigenstate)

will be $\quad \boxed{\Phi(\{\lambda\}) = B(\lambda_1) \dots B(\lambda_\ell)\, \Omega}$

- we will show that under certain conditions on the set $\{\lambda\}$, the Bethe Ansatz eqs., Φ will also be an eigenvector of $F(\lambda)$:

- $A(\lambda)\, B(\lambda_1) \dots B(\lambda_\ell)\, \Omega = \left(f(\lambda-\lambda_1) B(\lambda_1) A(\lambda) + g(\lambda-\lambda_1) B(\lambda) \overset{\Downarrow}{A(\lambda_1)} \right) \cdot B(\lambda_2) \dots B(\lambda_\ell)\, \Omega$

$$= \dots = \prod_{k=1}^{\ell} f(\lambda-\lambda_k)\, \overset{\text{from } A\Omega}{\alpha^N(\lambda)}\, B(\lambda_1) \dots B(\lambda_\ell)\, \Omega$$

$$+ \sum_{k=1}^{\ell} M_k(\lambda, \{\lambda\})\, B(\lambda_1) \dots \underset{\text{absent in product}}{\hat{B}(\lambda_k)} \dots B(\lambda_\ell)\, B(\lambda)\, \Omega$$

can be commuted through as all B's commute

$k=1$ is easy: contribution from $\Downarrow$ when then commuting A through all B's

$$\Rightarrow M_1(\lambda, \{\lambda\}) = g(\lambda-\lambda_1)\, \alpha^N(\lambda_1) \prod_{k=2}^{\ell} f(\lambda_1 - \lambda_k)$$

- due to the commutativity of all B's we could have considered $B(\lambda_j)$ instead of $B(\lambda_1)$ first $\Rightarrow M_j(\lambda, \{\lambda\})$ from $\lambda_1 \to \lambda_j$ in M_1

$$\Rightarrow M_j(\lambda, \{\lambda\}) = g(\lambda-\lambda_j)\, \alpha^N(\lambda_j) \prod_{\substack{k=1 \\ k \neq j}}^{\ell} f(\lambda_j - \lambda_k)$$

- likewise we consider D acting on Φ :

$$\Rightarrow D(\lambda) B(\lambda_1) \ldots B(\lambda_\ell) \Omega = \prod_{k=1}^{\ell} f^*(\lambda - \lambda_k) \delta^N(\lambda) B(\lambda_1) \ldots B(\lambda_\ell) \Omega$$

$$+ \sum_{k=1}^{\ell} N_k(\lambda, \{\lambda\}) B(\lambda_1) \ldots \hat{B}(\lambda_k) \ldots B(\lambda_\ell) B(\lambda) \Omega$$

where $N_j(\lambda, \{\lambda\}) = -g(\lambda - \lambda_j) \delta^N(\lambda_j) \prod_{\substack{k=1 \\ k\neq j}}^{\ell} f^*(\lambda_j - \lambda_k)$

- in order to obtain Φ as an eigenfunction of $F(\lambda) = A + D$ we need:

$$\boxed{\begin{aligned} &(A(\lambda) + D(\lambda)) \, \Phi(\{\lambda\}) = \Lambda(\lambda, \{\lambda\}) \, \Phi(\{\lambda\}) \\ \text{with} \quad &\Lambda(\lambda, \{\lambda\}) = \alpha^N(\lambda) \prod_{j=1}^{\ell} f(\lambda - \lambda_j) + \delta^N(\lambda) \prod_{j=1}^{\ell} f^*(\lambda - \lambda_j) \end{aligned}}$$

iff the terms with M_k's and N_k's cancel each other

This only happens if the set $\{\lambda\} = \{\lambda_1, \lambda_2, \ldots, \lambda_\ell\}$ satisfy

$$\forall_{j=1,\ldots,\ell} \quad \alpha^N(\lambda_j) \prod_{\substack{k=1 \\ k\neq j}}^{\ell} f(\lambda_j - \lambda_k) \overset{!}{=} \delta^N(\lambda_j) \prod_{\substack{k=1 \\ k\neq j}}^{\ell} f^*(\lambda_j - \lambda_k)$$

$$\Longleftrightarrow \forall_{j=1,\ldots,\ell} \quad \boxed{\frac{(\lambda_j + \frac{i}{2})^N}{(\lambda_j - \frac{i}{2})^N} = \prod_{\substack{k=1 \\ k\neq j}}^{\ell} \left(\frac{\lambda_j - \lambda_k - i}{\lambda_j - \lambda_k + i} \right)}$$

- these are the Bethe Ansatz eqs. (BAE) obtained in a different way by Hans Bethe 1931 in the solution of $XXX_{\frac{1}{2}}$. The method presented here generalizes to other spin chains!