Eigenvalues of Spin op. & Hamiltonian

- the FCR for T p. 146 imply commutation relations among Spin components $S^{\pm}, S^3$ and A, B, C, D:

  limit $\mu \to \infty$ FCR $\Rightarrow$ $$\boxed{\left[T_a(\lambda)\,,\ \tfrac{1}{2}\sigma_a^\alpha + \sum_n S_n^\alpha\right] = 0}$$

- expand $T_a(\mu) = \mu^N \underline{I} \otimes \underline{I}_a + i\mu^{N-1} \sum_{\alpha,n} S_n^\alpha \otimes \sigma_a^\alpha + \dots$

  and $R_{a_1a_2}(\lambda-\mu) = -\mu\, \underline{I}_{a_1} \otimes \underline{I}_{a_2} + \lambda\, \underline{I}_{a_1} \otimes \underline{I}_{a_2} + i P_{a_1,a_2}$

  to leading order and compare LHS with RHS of FCR:

$$\left((\lambda-\mu)\underline{I}_{a_1a_2} + iP_{a_1a_2}\right) T_{a_1}(\lambda) \left(\mu^N \underline{I} \otimes \underline{I}_{a_2} + i\mu^{N-1} \sum_{\alpha,n} S_n^\alpha \otimes \sigma_{a_2}^\alpha + \dots\right)$$

$$= \left(\mu^N \underline{I} \otimes \underline{I}_{a_2} + i\mu^{N-1} \sum_{\alpha,n} S_n^\alpha \otimes \sigma_{a_2}^\alpha + \dots\right) T_{a_1}(\lambda) \left((\lambda-\mu)\underline{I}_{a_1a_2} + iP_{a_1a_2}\right)$$

$\mathcal{O}(\mu^{N+1})$: $\quad -\underline{I}_{a_1a_2} T_{a_1}(\lambda)\, \underline{I} \otimes \underline{I}_{a_2} = -\underline{I} \otimes \underline{I}_{a_2} T_{a_1}(\lambda)\, \underline{I}_{a_1a_2}$ satisfied ($\underline{I}$'s commute)

$\mathcal{O}(\mu^N)$: $\quad \left(\lambda \underline{I}_{a_1a_2} + iP_{a_1a_2}\right) T_{a_1}(\lambda)\, \underline{I} \otimes \underline{I}_{a_2} - i\underline{I}_{a_1a_2} T_{a_1}(\lambda) \sum_{\alpha,n} S_n^\alpha \otimes \sigma_{a_2}^\alpha$

$$= \underline{I} \otimes \underline{I}_{a_2} T_{a_1}(\lambda) \left(\lambda \underline{I}_{a_1a_2} + iP_{a_1a_2}\right) + i \sum_{\alpha,n} S_n^\alpha \otimes \sigma_{a_2}^\alpha T_{a_1}(\lambda) \left(-\underline{I}_{a_1,a_2}\right)$$

$\Leftrightarrow$ $\quad iP_{a_1a_2} T_{a_1}(\lambda) - iT_{a_1}(\lambda) \sum_\alpha S^\alpha \otimes \sigma_{a_2}^\alpha = T_{a_1}(\lambda)\, iP_{a_1a_2} - i\sum_\alpha S^\alpha \otimes \sigma_{a_2}^\alpha T_{a_1}(\lambda)$

using $P_{a_1a_2} = \frac{1}{2} \underline{I}_{a_1} \otimes \underline{I}_{a_2} + \frac{1}{2} \sum_\alpha \sigma_{a_1}^\alpha \otimes \sigma_{a_2}^\alpha$ where the first part commutes

we have $0 = \left(\frac{1}{2} \sum_\alpha \sigma_{a_1}^\alpha \otimes \sigma_{a_2}^\alpha + \sum_\alpha S^\alpha \otimes \sigma_{a_2}^\alpha\right) T_{a_1}(\lambda)$

$\quad - T_{a_1}(\lambda) \left(\text{———} \ '' \ \text{———} \right)$

$$= \sum_\alpha \left(\tfrac{1}{2}\sigma_{a_1}^\alpha + S^\alpha\right) T_{a_1}(\lambda) \otimes \sigma_{a_2}^\alpha$$

$$- \sum_\alpha T_{a_1}(\lambda) \left(\tfrac{1}{2}\sigma_{a_1}^\alpha + S^\alpha\right) \otimes \sigma_{a_2}^\alpha$$

which has to be satisfied $\forall \alpha = 1, 2, 3$ in the P. (σ^α basis) 154

- Let us write this in matrix rep.:

$$T_a(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} , \quad S^{\pm} = S^1 \pm i S^2$$

- changing basis from $\alpha = 1,2,3$ to $\alpha = +,-,3$ we have

$S^\alpha + \frac{1}{2}\sigma_a^\alpha$:

$* \ \alpha = +$: $S^+ + \frac{1}{2}(\sigma_a^1 + i\sigma_a^2) = S^+ + \sigma_a^+ = \begin{pmatrix} S^+ & 0 \\ 0 & S^+ \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$* \ \alpha = -$: $S^- + \frac{1}{2}(\sigma_a^1 - i\sigma_a^2) = S^- + \sigma_a^- = \begin{pmatrix} S^- & 0 \\ 0 & S^- \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$* \ \alpha = 3$: $S^3 + \frac{1}{2}\sigma^3 = \begin{pmatrix} S^3 & 0 \\ 0 & S^3 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

$\alpha = +$

$$\Rightarrow 0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} S^+ & 1 \\ 0 & S^+ \end{pmatrix} - \begin{pmatrix} S^+ & 1 \\ 0 & S^+ \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} AS^+ & A+BS^+ \\ CS^+ & C+DS^+ \end{pmatrix} - \begin{pmatrix} S^+A+C & S^+B+D \\ S^+C & S^+D \end{pmatrix}$$

$$\Rightarrow \boxed{[S^+, C] = 0}, \quad \boxed{[S^+, B] = A - D} \quad [S^+, D] = C = -[S^+, A]$$

(ditto from $\alpha = -$: $[S^-, B] = 0$, $[S^-, C] = -(A-D)$, $[S^-, A] = B = -[S^-, D]$)

and
$$0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} S^3+\frac{1}{2} & 0 \\ 0 & S^3-\frac{1}{2} \end{pmatrix} - \begin{pmatrix} S^3+\frac{1}{2} & 0 \\ 0 & S^3-\frac{1}{2} \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A(S^3+\frac{1}{2}) & B(S^3-\frac{1}{2}) \\ C(S^3+\frac{1}{2}) & D(S^3-\frac{1}{2}) \end{pmatrix} - \begin{pmatrix} (S^3+\frac{1}{2})A & (S^3+\frac{1}{2})B \\ (S^3-\frac{1}{2})C & (S^3-\frac{1}{2})D \end{pmatrix}$$

$$\Rightarrow 0 = BS^3 - \tfrac{1}{2}B - S^3B - \tfrac{1}{2}B \Leftrightarrow \boxed{[S^3, B] = -B} \quad \text{etc.}$$

"ground state":
$$S_n^+ e^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \ \forall n \Rightarrow \boxed{S^+\Omega = \sum_n S_n^+ \bigotimes_{e=1}^{N} e_e^+ = 0}$$

and
$$S_n^3 e^+ = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \tfrac{1}{2} \ \forall n \Rightarrow \boxed{S^3\Omega = \sum_n S_n^3 \bigotimes_{e=1}^{N} e_e^+ = \frac{N}{2}\Omega}$$

$\Rightarrow \Omega$ has maximal 3-(z-) component and is highest weight "excited states":

- $\underline{S^3 \Phi(\{\lambda\})} = S^3 B(\lambda_1) \dots B(\lambda_\ell)\,\Omega = (B(\lambda_1) S^3 - B(\lambda_1))\, B(\lambda_2) \dots B(\lambda_\ell)\,\Omega$

$$= \dots = B(\lambda_1) \dots B(\lambda_\ell)(S^3 - \ell)\,\Omega = \underline{\left(\tfrac{N}{2} - \ell\right)\Phi} \quad \text{eigenstate}$$

- we'll show that Φ is also highest weight.

$$0 \overset{!}{=} S^+ \Phi = S^+ B(\lambda_1) \dots B(\lambda_\ell)\,\Omega = (B(\lambda_1) S^+ + A(\lambda_1) - D(\lambda_1)) B(\lambda_2) \dots B(\lambda_\ell)\,\Omega$$

$$= \sum_{j=1}^{\ell} B(\lambda_1) \dots B(\lambda_{j-1}) \left(A(\lambda_j) - D(\lambda_j)\right) B(\lambda_{j+1}) \dots B(\lambda_\ell)\,\Omega$$

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$$= \sum_{j=1}^{\ell} B(\lambda_1) \dots B(\lambda_{j-1}) \Big\{ \prod_{k=j+1}^{\ell} f(\lambda_j - \lambda_k)\, \alpha^N(\lambda_k)\, B(\lambda_{j+1}) \dots B(\lambda_\ell)\,\Omega$$

$$+ \sum_{k=j+1}^{\ell} g(\lambda_j - \lambda_k) \prod_{i \neq k}^{\ell} f(\lambda_k - \lambda_i)\, \alpha^N(\lambda_k)\, B(\lambda_{j+1}) \dots \hat{B}(\lambda_k) \dots B(\lambda_\ell)\,\Omega$$

$$- \prod_{k=j+1}^{\ell} f^*(\lambda_j - \lambda_k)\, \delta^N(\lambda_k)\, B(\lambda_{j+1}) \dots B(\lambda_\ell)\,\Omega$$

$$+ \sum_{k=j+1}^{\ell} g(\lambda_j - \lambda_k) \prod_{i \neq k}^{\ell} f^*(\lambda_k - \lambda_i)\, \delta^N(\lambda_k)\, B(\lambda_{j+1}) \dots \hat{B}(\lambda_k) \dots B(\lambda_\ell)\,\Omega \Big\}$$

- vanishes if $\{\lambda_j\}$ satisfy BAE
- Φ highest weight $\Rightarrow$ S^3 eigenvalue non-negative $\Rightarrow$ $\boxed{\ell \leq \frac{N}{2}}$

Hamiltonian:

shift operator: $U \Phi = i^{-N} F(\lambda = \tfrac{i}{2})\, \Phi = i^{-N} (A(\tfrac{i}{2}) + D(\tfrac{i}{2}))\, \Phi$

$$\overset{\text{ev.}}{=} i^{-N} \Big(\alpha^N(\tfrac{i}{2}) \prod_{j=1}^{\ell} f(\tfrac{i}{2} - \lambda_j) + \underbrace{\delta^N(\tfrac{i}{2})}_{0} \prod_{j=1}^{\ell} f^*(\tfrac{i}{2} - \lambda_j) \Big) \Phi$$

$$= \cancel{i^{-N} (\tfrac{i}{2} + \tfrac{i}{2})^N} \prod_{j=1}^{\ell} \frac{(\lambda_j + \frac{i}{2})}{(\lambda_j - \frac{i}{2})}\, \Phi$$

from p. 148 $X_n U = U X_{n-1}$ we have $U = \exp[iP]$, P momentum

$\Rightarrow$ taking the logarithm we have for $\underline{P}$ acting on $\underline{\Phi}$:

$$\underline{P}\,\underline{\Phi} = i^{-1}\ln U\,\phi = \sum_{j=1}^{\ell} \underbrace{i^{-1}\ln\left|\frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}}\right|}_{\equiv p(\lambda_j)} \phi \equiv \left(\sum_{j=1}^{\ell} p(\lambda_j)\right)\underline{\Phi}$$

additivity of each indiv. momentum

we have $H = \frac{i}{2}\frac{d}{d\lambda}\ln F(\lambda)\Big|_{\lambda=\frac{i}{2}} - \frac{N}{2}$

$$\Rightarrow \boxed{H\underline{\Phi}} = \left\{\frac{i}{2F(\frac{i}{2})}\left(\underbrace{\left(\frac{d}{d\lambda}\alpha(\lambda)\right)}_{N\frac{F(\frac{i}{2})}{(\frac{i}{2})}}\Pi f + \alpha^N(\lambda)\frac{d}{d\lambda}\Pi f\right)\Big|_{\lambda=\frac{i}{2}} - \frac{N}{2}\right\}\underline{\Phi}$$

$$= \frac{i}{2}\sum_{j=1}^{\ell}\frac{f'(\lambda-\lambda_j)}{f(\lambda-\lambda_j)}\Big|_{\lambda=\frac{i}{2}}\underline{\Phi} = \frac{i}{2}\sum_{j=1}^{\ell}\left(\frac{1}{\frac{i}{2}-\lambda_j-i} - \frac{1}{\frac{i}{2}-\lambda_j}\right)\underline{\Phi} = \boxed{\left(-\frac{1}{2}\right)\sum_{j=1}^{\ell}\frac{1}{\lambda_j^2+\frac{1}{4}}\,\underline{\Phi}}$$

with additive energy eigenvalues $\boxed{\epsilon(\lambda) = -\frac{1}{2}\frac{1}{\lambda^2+\frac{1}{4}} \overset{a}{=} \frac{1}{2}p(\lambda)'}$

$\to$ all energies are <0, so

Ω is <u>not</u> the ground state of H (but of $-H$!)

$-H$ is a ferromagnet ($+H$ = antiferromagnet)

- every <u>quasiparticle state</u> created by $B(\lambda_j)$ acting on Ω

  - decreases S^3 by $+1$
  - has energy $\epsilon(\lambda_j)$ and momentum $p(\lambda_j)$

  λ is called <u>rapidity</u> of a quasiparticle

The thermodynamical limit $N \to \infty$

- we will only briefly consider an example here, choosing $-H$ so that Ω is indeed the ground state

- consider $$N p(\lambda_j) = \frac{N}{i} \ln \frac{(\lambda_j + \frac{i}{2})}{(\lambda_j - \frac{i}{2})} \overset{\text{BAE}}{=} \frac{1}{i} \ln \left[\prod_{\substack{k=1 \\ k \neq j}}^{\ell} \frac{(\lambda_j - \lambda_k + i)}{(\lambda_j - \lambda_k - i)} \right]$$

- in general the argument of the ln will be complex

$$\Rightarrow \quad \operatorname{Log} z = \underbrace{\log|z| + i \arg(z)}_{\text{principle branch}} + 2\pi i k \quad , k \in \mathbb{Z}$$

$\sharp$ branches of log in $\mathbb{C}$

fixing these branches we have

$$N p(\lambda_j) = 2\pi q_j + \sum_{\substack{k=1 \\ k \neq j}}^{\ell} \underbrace{\ln \left(\frac{\quad}{\quad} \right)}_{\equiv \varphi(\lambda_j - \lambda_k)}$$

log at fixed branch

$\Rightarrow$ for $N \to \infty$, ℓ fixed we get to leading order

$$\boxed{p(\lambda_j) = \frac{2\pi q_j}{N}} + \text{corrections through scattering.}$$

- the φ's play the rôle of phase shifts

Let us consider the BAE for $\ell = 2$ as an explicit example:

$$\left(\frac{\lambda_1 + \frac{i}{2}}{\lambda_1 - \frac{i}{2}} \right)^N = \frac{(\lambda_1 - \lambda_2 + i)}{(\lambda_1 - \lambda_2 - i)} \quad , \quad \left(\frac{\lambda_2 + \frac{i}{2}}{\lambda_1 - \frac{i}{2}} \right)^N = \frac{\lambda_2 - \lambda_1 + i}{\lambda_2 - \lambda_1 - i} = \frac{(\lambda_1 - \lambda_2 - i)(-1)}{(\lambda_1 - \lambda_2 + i)(-1)}$$

$$\Rightarrow \quad \boxed{\left(\frac{\lambda_1 + \frac{i}{2}}{\lambda_1 - \frac{i}{2}} \right)^N \left(\frac{\lambda_2 + \frac{i}{2}}{\lambda_1 - \frac{i}{2}} \right)^N = 1}$$

note that solutions to $w^N = 1$, $w \in \mathbb{C}$ are roots of unity $w_k = e^{2\pi i k / N}$, $k = 0, 1, \ldots, N-1$

- now let us solve the BAE at $N \to \infty$:

for $z = r e^{i\varphi}$: $z^N = r^N e^{iN\varphi} \xrightarrow[N\to\infty]{} \begin{cases} 0 & r<1 \\ \infty & r>1 \end{cases}$

- this behaviour can only be satisfied if in the BAE the rhs also becomes 0 or ∞ for $N \to \infty$

one can show that a solution at $N \to \infty$ is

$$\boxed{\lambda_1 = \lambda_{\frac{1}{2}} + \frac{i}{2}, \quad \lambda_2 = \lambda_{\frac{1}{2}} - \frac{i}{2}}, \text{ with } \lambda_{\frac{1}{2}} \in \mathbb{R}\ (\setminus \{0\})$$

$\Rightarrow$ $\underline{j=1}$: LHS $\left(\frac{\lambda_{\frac{1}{2}}+i}{\lambda_{\frac{1}{2}}}\right)^N = \left(1+\frac{i}{\lambda_{\frac{1}{2}}}\right)^N = \underbrace{\left(1+\frac{1}{\lambda_{\frac{1}{2}}^2}\right)}_{>1}^{\frac{N}{2}} e^{i\varphi N} \to \infty$

and RHS is $\frac{2i}{0}$

$\underline{j=2}$: LHS $\left(\frac{\lambda_{\frac{1}{2}}}{\lambda_{\frac{1}{2}}-i}\right)^N = \left(\frac{\lambda_{\frac{1}{2}}(\lambda_{\frac{1}{2}}+i)}{\lambda_{\frac{1}{2}}^2+1}\right)^N = \left(\frac{\lambda_{\frac{1}{2}}^2(\lambda_{\frac{1}{2}}^2+1)}{(\lambda_{\frac{1}{2}}^2+1)^2}\right)^{\frac{N}{2}} e^{iN\varphi} \to 0$

and LHS is $\frac{0}{-2i}$ $\qquad \frac{1}{1+\frac{1}{\lambda_{\frac{1}{2}}^2}} < 1$

- because of $\lambda_j \in \mathbb{C}$ the energy eigenvalues p 157 $\epsilon(\lambda_j)$ will have a real and imag. part $\to \nexists$ bound states.

However the total momentum is $\in \mathbb{R}$: $p(\lambda_1)+p(\lambda_2) = \frac{1}{i} \ln \underbrace{\left(\frac{\lambda_1+i}{\lambda_2-i}\right)\left(\frac{\lambda_2+i}{\lambda_1-i}\right)}_{\text{root of unity } e^{i\frac{K}{N}}} = \frac{K}{N} \in \mathbb{R}$

the ground state energy is obtained from that

$\underline{P}(\lambda_{\frac{1}{2}}) = p(\lambda_{\frac{1}{2}}+\frac{i}{2}) + p(\lambda_{\frac{1}{2}}-\frac{i}{2}) = \frac{1}{i} \ln\left(\frac{\lambda_{\frac{1}{2}}+i}{\lambda_{\frac{1}{2}}-i}\right)$

$E(\lambda_{\frac{1}{2}}) = \frac{1}{2}\frac{d}{d\lambda} \underline{P}(\lambda_{\frac{1}{2}}) = \frac{-1}{\lambda_{\frac{1}{2}}^2+1} \in \mathbb{R}$ energy of $-H$

- the ground state the SU(2) symmetry at $N \to \infty$ = ferromagnetic phase

more $\to$ notes by Faddeev

Yang-Baxter eq:

reorder 3 T's : $T_{a_1}(\lambda_1)\, T_{a_2}(\lambda_2)\, T_{a_3}(\lambda_3)$ on $V_1 \otimes V_2 \otimes V_3$

$$\equiv T_1 \quad T_2 \quad T_3$$

write FCR for T (p. 146) as

$$R_{12} T_1 T_2 = T_2 T_1 R_{12}$$

- there are 2 possibilities to permute :

$$\begin{cases} 123 \to 213 \to 231 \to 321 \\ \text{or } 123 \to 132 \to 312 \to 321 \end{cases}$$

$$R_{23} R_{13} R_{12} T_1 T_2 T_3 = R_{23} R_{13} T_2 T_1 R_{12} T_3 = R_{23} T_2 R_{13} T_1 T_3 R_{12}$$
$$= R_{23} T_2 T_3 T_1 R_{13} R_{12} = T_3 T_2 R_{23} T_1 R_{13} R_{12}$$
$$= T_3 T_2 T_1 R_{23} R_{13} R_{12}$$

or

$$R_{12} R_{13} R_{23} T_1 T_2 T_3 = R_{12} R_{13} T_1 T_3 T_2 R_{23} = R_{12} T_3 T_1 T_2 R_{13} R_{23}$$
$$= T_3 T_2 T_1 R_{12} R_{13} R_{23}$$

this is only consistent if

$$\boxed{R_{a_1a_2}(\lambda_1-\lambda_2)\, R_{a_1a_3}(\lambda_3-\lambda_2)\, R_{a_2a_3}(\lambda_2-\lambda_3) = R_{a_2a_3}(\lambda_2-\lambda_3)\, R_{a_1a_3}(\lambda_1-\lambda_3)\, R_{a_1a_2}(\lambda_1-\lambda_2)}$$

Yang-Baxter eq. (CN Yang PRL 1967, R.J. Baxter 1972)

- this and the algebraic Bethe Ansatz is the starting point to study quantum integrable systems in statistical mechanics

- there is a classical limit of YBE in terms of commutators, which can be used to classify classical integrable systems, using Lie alg. In the case of quantum integrability this leads to quantum groups