The KdV eq. as a Hamiltonian System

Q: can we find a Hamiltonian $H[u]$ and a PB $\{,\}$ for the field $u(x)$ s.th. $\partial_t u(x) \overset{!}{=} \{u(x), H[u]\}_2 = u\partial_x u(x) + \partial_x^3 u(x)$?

a) try $H_2[u] = \int_{-\infty}^{\infty} dx \left(\frac{1}{3!} u^3(x) - \frac{1}{2} (\partial_x u(x))^2 \right)$

"with canonical" PB $\{u(x), u(y)\}_2 \equiv \partial_x \delta(x-y) = f_2(x,y)$, b.c. $u, u', \ldots \to 0$ for $x \to \pm\infty$

=> using the previous example of the Harmonic oscillator $f^{-1}(x,y) = G(x-y)$

<u>check that we get the KdV:</u>

$$\dot{u}(x) = \{u(x), H_2[u]\}_2 = \int_{-\infty}^{\infty} dz\, \partial_x \delta(x-z) \frac{\delta H_2[u]}{\delta u(z)}$$

$$= \int_{-\infty}^{\infty} dz\, \partial_x \delta(x-z) \left[\frac{1}{2} u^2(z) - \int_{-\infty}^{\infty} dy\, \partial_y u(y)\, \partial_y \delta(y-z) \right]$$

$$= \int_{-\infty}^{\infty} dz \left(-\partial_z \delta(x-z)\right) \left[\frac{1}{2} u^2(z) + \partial_z^2 u(z) \right]$$

int. by parts

$$= u \partial_x u(x) + \partial_x^3 u(x) \qquad \text{KdV} \checkmark$$

<u>Note:</u> $\exists$ other Hamiltonians and other PB leading to KdV!

- this is a typical property of integrable systems
- given $H_1, H_2[u]$ leading both to KdV they are different (indep) if $\{H_1, H_2\} = 0$
- $H_2 = H_2[u(x)]$; $\not\exists$ not explicitly t-dep. => $\frac{dH_2}{dt} = 0$ ($= \frac{\partial H_2}{\partial t}$) conserved as $\{H_2, H_2\}_2 = 0$

b) second realisation

$$H_1[u] = \int_{-\infty}^{\infty} dx\, \frac{1}{2} u^2(x) \qquad \left(\text{again no explicit } t\text{-dep} \Rightarrow \frac{dH_1}{dt} = 0\right)$$

$\frac{\partial u}{\partial t_1}$

conserved

with new P.B.

$$\{u(x), u(y)\}_1 \equiv \left(\partial_x^3 + \frac{1}{3}(\partial_x u(x) + u(x)\partial_x)\right)\delta(x-y) \equiv f_1(x,y)$$

acting on both $u(x)$ and $\delta(x-y)$ (everything to the right)

check:

$$\dot{u}(x) = \{u(x), H_1[u]\}_1 = \int_{-\infty}^{\infty} dz\, f_1(x,z) \underbrace{\frac{\delta H[u]}{\delta u(z)}}_{= u(z)}$$

$$= \int_{-\infty}^{\infty} dz \left(\partial_x^3 + \frac{1}{3}(\partial_x u(x) + u(x)\partial_x)\right)\delta(x-z)\, u(z)$$

$$= \partial_x^3 u(x) + \frac{1}{3}\left(\partial_x u^2(x) + u\,\partial_x u(x)\right) = \partial_x^3 u(x) + u\,\partial_x u(x)$$

KdV ✓

Note: the differential operator acting in $f_1(x,y)$ can be integrated by parts (check this

Q: a) Interpretation of H_1, H_2?

b) Are they independent?

c) Are there other conserved quantities? (independent?)

a) Both H_1, H_2 are conserved quantities,

Noether Theorem $\Rightarrow$ generate sym. trafos to a) • choose H_2 as Hamiltonian, $\{,\}_2$ as P.B.

$\Rightarrow \quad \{u(x), H_2\}_2 = \partial_t u(x)$ $\quad$ H_2 generates time translation (well known for Hamiltonian)

• $\{u(x), H_1\}_2 = \int_{-\infty}^{\infty} dy\, f_2(x,y) \frac{\delta H_1[u]}{\delta u(y)} = \int_{-\infty}^{\infty} dy\, \partial_x \delta(x-y)\, u(y)$

$= \partial_x u(y)$ $\quad$ H_1 generates space-translation, like a conserved momentum would do

to b) • $\frac{dH_1}{dt} = 0$ is conserved (and not explicitly t-dep)

$\Rightarrow \quad \{H_1, H_2\}_2 = 0$ $\quad$ so H_1 & H_2 are independent

(check this using the definition)

to c) Yes: choose for example $H_0 = 3 \int_{-\infty}^{\infty} dx\, u(x)$ $\quad$ convention, for later

and it holds $\{H_i, H_j\}_2 = 0 \quad \forall i,j = 0,1,2$

H_j not explicitly t-dep $\left(\frac{\partial H_j}{\partial t} = 0\right) \Rightarrow \frac{dH_j}{dt} = 0$ conserved

Note: later we will construct an infinite tower of H_n's we have $\quad \frac{\delta H_0}{\delta u(x)} = 3, \quad \frac{\delta H_1}{\delta u(x)} = u(x), \quad \frac{\delta H_2}{\delta u(x)} = \frac{1}{2}u(x)^2 + \partial_x^2 u(x)$

$\Rightarrow$ for $\{H_i, H_j\}_2 = \int dx\,dy \, \frac{\delta H_i}{\delta u(x)} \, J_2(x,y) \, \frac{\delta H_j}{\delta u(y)}$

- $\{H_0, H_1\}_2 = \int\limits_{-\infty}^{\infty} dx\,dy \; 3 \; \partial_x \delta(x-y) \, u(y) - \int\limits_{-\infty}^{\infty} dx \, \partial_x u(x) \overset{b.c.}{=} 0$

- $\{H_0, H_2\}_2 = \int\limits_{-\infty}^{\infty} dx\,dy \; 3 \, \partial_x \delta(x-y) \left[\frac{1}{2} u(y)^2 + \partial_y^2 u(y)\right]$

$$= 3 \int\limits_{-\infty}^{\infty} dx \, \partial_x \left[\frac{1}{2} u(x)^2 + \partial_x^2 u(x)\right] \overset{b.c.}{=} 0$$

- exercise $\quad \{H_1, H_2\}_2 = 0$

Q: Is there also a Lagrange formulation of the KdV?

$\rightarrow$ Let us first construct a continuous Lagrange form of the Harmonic Oscillator:

discrete: $\mathcal{H}(p,q) = \sum_{j=1}^{n} \left(\frac{1}{2} p_j^2 + \frac{1}{2}\omega^2 q_j^2\right) \quad , \quad p_j \equiv \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}_j}$
(m=1)

follows from $\quad \mathcal{L} = \sum_{j=1}^{n} \left(\frac{1}{2}\dot{q}_j^2 - \frac{\omega^2}{2} q_j^2\right) \quad \Rightarrow \quad p_j = \dot{q}$

e.o.m. $\qquad 0 = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i}$

as $\quad \mathcal{H}(p,q) = \sum_j \dot{q}_j p_j - \mathcal{L}(q,\dot{q},t) = \sum_j \frac{1}{2}\dot{q}_j^2 + \frac{1}{2}\omega^2 q_j^2$

for $\mathcal{H}$ we know how to construct the continuous form & its e.o.m p.18: $\mathcal{H}[u] = \int_{-\infty}^{\infty} dx \, \frac{1}{2}\omega u(x)^2$, e.o.m $\boxed{\dot{u}(x) = \int_{-\infty}^{\infty} dy \, \epsilon(x-y)\omega u(y)}$

$\overset{\partial_x}{\Longrightarrow}$ $\underline{\partial_x\partial_t u(x) = \omega u(x)}$

P.B. $\{u, \mathcal{H}\} = \int dy \underbrace{f(x,y)}_{\substack{\shortparallel \\ \epsilon(x-y)}} \frac{\delta\mathcal{H}}{\delta u}$

Construction of $I[u]$:

Ansatz

(need f^{-1} to construct I)

$$I[u] = \frac{1}{2}\int_{-\infty}^{\infty} dx dy \, u(x) \underbrace{f^{-1}(x,y)}_{\substack{\partial_x\delta(x-y) \\ \shortparallel}} \partial_t u(y) - \underbrace{\int dx \, \frac{\omega}{2} u^2(x)}_{\mathcal{H}[u]}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} dx \left[u(x)\partial_x\partial_t u(x) - \omega u(x)^2 \right]$$

check if we get the correct e.o.m. from Euler-Lagrange:

($u, \dot{u}$ indep)

$$0 = \partial_t \frac{\delta I}{\delta \dot{u}(x)} - \frac{\delta I}{\delta u x}$$

with $\frac{\delta I}{\delta u(x)} = \frac{1}{2}\partial_x\partial_t u(x) - \omega u(x)$

$\partial_t \frac{\delta I}{\delta \dot{u}(x)} = -\frac{1}{2}\partial_t(\partial_x u(x))$, suppose $u(x)$ & partial derival smooth

$$\Rightarrow 0 = -\frac{1}{2}\partial_t\partial_x u - \frac{1}{2}\partial_x\partial_t u + \omega u(x)$$

$$\Leftrightarrow \partial_x\partial_t u(x) = \omega u(x)$$

- follow the same construction of I starting from $\mathcal{H}$-KdV (there role of f, f^{-1} interchanged for $\{,\}_2$)

<3

$$H_2 = \int_{-\infty}^{+\infty} dx \left(\frac{1}{3!} u^3(x) - \frac{1}{2} (\partial_x u(x))^2 \right), \qquad f_2(x,y) = \partial_x \delta(x-y)$$

$$f_2^{-1}(x,y) = \epsilon(x-y)$$

$$\Rightarrow \mathcal{L}_{kdV} = \frac{1}{2} \int_{-\infty}^{\infty} dx\, dy\, u(x)\, f_2^{-1}(x,y)\, \partial_t u(y) - H_2[u]$$

$$\boxed{\mathcal{L}_{kdV} = \frac{1}{2} \int_{-\infty}^{\infty} dx\, dy\, u(x)\, \epsilon(x-y)\, \partial_t u(y) - \int_{-\infty}^{\infty} dx \left(\frac{1}{3!} u^3(x) - \frac{1}{2} (\partial_x u(x))^2 \right)}$$

check that Euler-Lagrange gives kdV: ($u, \dot{u}$ indep.!)

$$\partial_t \frac{\delta \mathcal{L}_{kdV}}{\delta \dot{u}(z)} = \partial_t \frac{1}{2} \int_{-\infty}^{\infty} dx\, u(x)\, \epsilon(x-z) \underset{\text{c.o.m}}{=} \boxed{\frac{1}{2} \int_{-\infty}^{\infty} dx\, \partial_t u(x)\, \epsilon(x-z)}$$

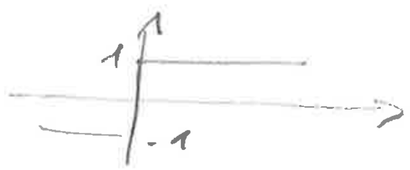
$$\frac{\delta \mathcal{L}_{kdV}}{\delta u(z)} = \boxed{\frac{1}{2} \int_{-\infty}^{\infty} dy\, \epsilon(z-y)\, \partial_t u(y) - \left(\frac{1}{2} u(z)^2 + \partial_z^2 u(z) \right)}$$

$$\partial_z \text{ of } \left(\partial_t \frac{\delta \mathcal{L}_{kdV}}{\delta \dot{u}(z)} - \frac{\delta \mathcal{L}_{kdV}}{\delta u(z)} = 0 \right) \quad \text{gives}$$

$$-\frac{1}{2} \partial_t u(z) - \frac{1}{2} \partial_t u(z) + u(z) \partial_z u(z) + \partial_z^3 u(z) = 0$$

kdV ✓

<u>Note</u> that because of $\epsilon(x-y)$



the first term in $\mathcal{L}_{kdV}$ is <u>non-local</u>! (for H.O. $\mathcal{L}$ <u>is</u> local)

- it turns out that no local $\mathcal{L}$ exists for the kdV equation

Recursive construction of conserved quantities for the KdV

- we are looking for conserved $H_n,\ n\in\mathbb{N}$:

$$\frac{dH_n}{dt} \overset{\text{no explicit } t\text{-dep}}{=} \{H_n, H_2\}_2 = 0$$

such that they are mutually independent

$$\{H_n, H_m\}_2 = 0 \quad \forall n,m\in\mathbb{N} \quad \text{in involution}$$

we already know:

Symmetry

$$H_0 = 3\int_{-\infty}^{\infty} dx\, u(x), \quad \frac{\delta H_0}{\delta u} = 3 \qquad \{u(x), H_0\}_2 = 0$$

$$\text{momentum } H_1 = \int_{-\infty}^{\infty} dx\, \tfrac{1}{2} u(x)^2, \quad \frac{\delta H_1}{\delta u} = u \qquad \{u(x), H_1\}_2 = \partial_x u \quad \text{translation in space}$$

$$\text{Hamiltonian } H_2 = \int_{-\infty}^{\infty} dx \left(\tfrac{1}{3!} u^3(x) - \tfrac{1}{2}(\partial_x u(x))^2\right) \qquad \{u, H_2\}_2 = \partial_t u \quad \text{translat. in time}$$

which are conserved and in involution

define recursion
$$\boxed{\left(\partial_x^3 + \tfrac{1}{3}(\partial_x u(x) + u(x)\partial_x)\right)\frac{\delta H_m[u]}{\delta u(x)} = \partial_x \frac{H_{m+1}[u]}{\delta u(x)}}$$

check $m=0$:
$$\tfrac{1}{3}(\partial_x u)\cdot 3 = \partial_x u(x) \quad ✓$$

check $m=1$:
$$\partial_x^3 u + u\partial_x u \overset{!}{=} \partial_x\left(\tfrac{1}{2}u(x)^2 + \partial_x^2 u(x)\right) \quad ✓$$

- explains why $\exists$ 2 PB $\{\}_{1,2}$ for KdV

$$\partial_t u = \{u(x), H_2\}_2 \overset{\{\}_2}{=} \partial_x \frac{\delta H_2}{\delta u(x)} \quad \text{KdV}$$

$$\partial_t u = \{u(x), H_1\}_1 \overset{\{\}_1}{=} \left(\partial_x^3 + \tfrac{1}{3}(\partial_x u + u\partial_x)\right)\frac{\delta H_1}{\delta u(x)} \quad \text{KdV}$$

Explicit construction of the H_n

given that we have a conserved quantity $Q[u]$ that is not explicitly t-dependent:

$$\frac{dQ}{dt} = \{Q[u], H_2\}_2 + \overset{=0}{\frac{\partial Q}{\partial t}} = 0$$

define a density ρ : $Q[u] = \int_{-\infty}^{\infty} dx\, \rho[u(x,t)]$

$$0 = \frac{dQ}{dt} = \int_{-\infty}^{\infty} dx \frac{\partial}{\partial t} \rho[u(x,t)] \quad \Rightarrow \quad \partial_t \rho \text{ has to vanish}$$

up to a total ∂_x-derivative:

$$\boxed{\partial_t \rho[u(x,t)] + \partial_x j[u(x,t)] = 0}$$

Continuity equation (with b.c. $u[x,t] \underset{x \to \pm\infty}{\to} 0$, ditto ρ & j)

- we will now construct explicitly the densities ρ_n s.th.

$$H_n[u] = \int_{-\infty}^{\infty} dx\, \rho_n[u]$$

$\rightarrow$ these are nontrivial only if $\rho_n \neq \partial_x \hat{\rho}_n$

Example ρ_0 : we have from KdV

$$\partial_t u = u \partial_x u + \partial_x^3 u = \partial_x\left(\tfrac{1}{2} u^2\right) + \partial_x\left(\partial_x^2 u\right)$$

read as continuity equation with

$$\tilde{\rho}_0[u(x,t)] = u(x,t], \quad j_0[u] = -\tfrac{1}{2} u^2 - \partial_x^2 u$$

define $Q_0 = H_0 = \int_{-\infty}^{\infty} dx\, 3\tilde{g}_0[u(x,t)]$ which is a constant of motion (conserved quant.)

(we knew this already, p 25 $\{u(x), H_0\}_2 = 0$)

Example g_1: starting from KdV $\cdot\, u$ we have

$$u\,\partial_t u = u\left(u\partial_x u + \partial_x^3 u\right)$$

$$g_1[u] \equiv \tfrac{1}{2}\partial_t u^2 =$$

- the rhs can be written as a total derivative (exercise!) and we obtain the known conserved quant.

$$H_1 = \int_{-\infty}^{\infty} dx\, g_1[u] = \int_{-\infty}^{\infty} dx\, \tfrac{1}{2} u(x,t)^2$$

Example g_2:

the Hamiltonian H_2 itself is conserved ; $\{H_2, H_2\}_2 = 0$

$$H_2 = \int_{-\infty}^{\infty} dx \left(\tfrac{1}{3!} u(x,t)^3 - \tfrac{1}{2}(\partial_x u)^2\right) \equiv \int_{-\infty}^{\infty} dx\, g_2[u]$$

Q: systematic construction of higher order g_n's? up to $n=11$ was done by hand.

Trick: we'll use a non linear map $u = f[v]$

yielding a modified KdV for v

- v will satisfy a continuity equation
- inverting $v = v[u]$ will yield the H_n

The Miura Transformation and modified KdV

KdV $\quad \partial_t u = u \partial_x u + \partial_x^3 u$

mod. KdV $\quad \partial_t v = v^2 \partial_x v + \partial_x^3 v$

by using the Riccati trafo $\quad \boxed{u(x,t) = v^2(x,t) + i\sqrt{6}\, \frac{\partial v}{\partial x}}$

Exercise: insert into KdV $\Rightarrow$ $D\left(\partial_t v - v^2 \partial_x v - \partial_x^3 v\right) = 0$

with D differential operator (funct. of v)

Conclusion: when v solves the mKdV the KdV in variables u also holds: $v \Rightarrow u$ using Riccati

Symmetries:

while the KdV is invariant under Galilei trafo

$$t \to t$$
$$x \to x + ct \qquad (*)$$
$$u \to u + c$$

the mKdV is not invariant under Galilei trafo

(exercise: check with $v \to v + c$ in mKdV)

- We now do a second Galilei trafo to obtain a second modification of KdV that depends on a parameter ε

trafo
$$t \to t$$
$$x \to x + \frac{3}{2\varepsilon^2} t \quad \Rightarrow \quad \partial_t \to \partial_t + \frac{3}{2\varepsilon^2}\partial_x$$
$$v \to \frac{\varepsilon}{\sqrt{6}} v + \frac{\sqrt{6}}{2\varepsilon} = \frac{\varepsilon}{\sqrt{6}}\left(v + \frac{3}{\varepsilon^2}\right)$$
$$u \to u + c$$

$\Rightarrow$ mapped mKdV $\quad \partial_t V = V^2 \partial_x V + \partial_x^3 V$

$$\to \quad \cancel{\frac{\varepsilon}{\sqrt{6}}}\left(\partial_t V + \frac{3}{2\varepsilon^2}\partial_x V\right) = \left(\frac{\varepsilon}{\sqrt{6}}V + \frac{\sqrt{6}}{2\varepsilon}\right)^2 \cancel{\frac{\varepsilon}{\sqrt{6}}}\partial_x V + \cancel{\frac{\varepsilon}{\sqrt{6}}}\partial_x^3 V$$

$$\Leftrightarrow \quad \partial_t V + \frac{3}{2\varepsilon^2}\partial_x V = \left(\frac{\varepsilon^2}{6}V^2 + V + \frac{3}{2\varepsilon^2}\right)\partial_x V + \partial_x^3 V$$

$$\Leftrightarrow \quad \boxed{\partial_t V = \left(\frac{\varepsilon^2}{6}V^2 + V\right)\partial_x V + \partial_x^3 V} \quad (1)$$

- interpolates between KdV $\varepsilon = 0 \Rightarrow \underline{u = V}$
  and mKdV $\varepsilon \to \infty$ rescaling $V \to \frac{\varepsilon}{\sqrt{6}}V$
  $\Rightarrow u = V^2 + i\sqrt{6}\,\partial_x V$

- (1) also obeys a continuity eq

$$\underline{\partial_t V = \partial_x\left(\frac{\varepsilon^2}{18}V^3 + \frac{1}{2}V^2 + \partial_x^2 V\right)}$$

and can be used to construct ∞-many conserved quant.

expand $\quad \underline{V[u,\varepsilon] = \sum_{n=0}^{\infty} \varepsilon^n V_n[u]}$ all conserved! (assume absolute convergence)

the trafo leading to (1) (Riccati + Galilei) was

$$u \to u + \frac{3}{2\varepsilon^2} = \frac{\varepsilon^2}{6}V^2 + V + \frac{3}{2\varepsilon^2} + i\sqrt{6}\frac{\varepsilon}{\sqrt{6}}\partial_x V$$

$$\Leftrightarrow \quad \boxed{u = V + \frac{\varepsilon^2}{6}V^2 + i\varepsilon\partial_x V}$$

insert the expansion into this trafo:

$$u(x,t) = \sum_{n=0}^{\infty} \varepsilon^n V_n + i \sum_{n=0}^{\infty} \varepsilon^{n+1} \partial_x V_n + \underbrace{\left(\sum_{n=0}^{\infty} \varepsilon^n V_n\right)^2}_{\sum_{n=0}^{\infty} \sum_{m=0}^{n} \varepsilon^m \varepsilon^{n-m} V_m V_{n-m}} \frac{\varepsilon^2}{6}$$

$$= \sum_{n=0}^{\infty} \varepsilon^n \left[V_n + i \partial_x V_{n-1} + \frac{1}{6} \sum_{m=0}^{n-2} V_{n-2-m} V_m \right]$$

where we have defined $V_{-1} \equiv 0$

and $\sum_{m=0}^{-2}$, $\sum_{m=0}^{-1}$ void

Examples: $n=0$: $u = V_0$ $\Rightarrow$ $H_0 = \int_{-\infty}^{\infty} dx \, 3u$ conserved,

$n=1$: $V_1 + i\partial_x V_0 = 0$

$\Leftrightarrow$ $V_1 = -i\partial_x u$, $\int_{-\infty}^{\infty} dx (-i\partial_x u)$ is trivially conserved

$n=2$: $V_2 + i\partial_x V_1 + \frac{1}{6} V_0^2 = 0$

$\Leftrightarrow$ $V_2 = -\frac{1}{6}u^2 - \partial_x^2 u$,

$$-3\int_{-\infty}^{\infty} dx \, V_2 = +\int_{-\infty}^{\infty} dx \left(\frac{3}{6}u^2 + \cancel{3\partial_x^2 u}\right) = \int_{-\infty}^{\infty} dx \, \frac{1}{2} u^2 = H_1$$

tot. derivative

is conserved

- one can show that all odd coeff.

$V_{2n+1} = i\partial_x (\;)$ are trivially conserved

(from Ansatz $v = z + iy$ and all V_{2n} are real, exercise) define the set $H_n[u] = 3(-)^n \int_{-\infty}^{\infty} dx\, V_{2n}[u] \quad \forall n \in \mathbb{N}$

- the V_{2n} may still contain total derivatives, we are only interested in the non-trivial contribution

<u>Independence of H_n : scaling behaviour :</u>

we have seen for

KdV inv. under scaling | mKdV inv if

$$\left.\begin{array}{lll} u \to c^{-2} u & \dim [u] = -2 \\ x \to c\, x & [x] = +1 \\ t \to c^3 t & [t] = +3 \end{array}\right\} \Rightarrow \quad \begin{array}{l} [V] = -2 \\ [\varepsilon] = +1 \end{array}$$

check as an exercise!

- $V = \sum_{n=0}^{\infty} \varepsilon^n V_n \quad \Rightarrow \quad [V_n] = -n-2$

$\Rightarrow$ all $H_n = 3(-)^n \int_{-\infty}^{\infty} dx\, V_{2n}[u(x,t)]$ scale differently

$$\underline{[H_n] = -2n-2-1}$$

- $\exists$ countably many conserved quantities with different scaling behaviour

- we will show that $\{H_n, H_m\}_2 = 0$

using a recurrance relation $H_n \Leftrightarrow H_{n+1}$

or $V_{2n} \Leftrightarrow V_{2n+2}$

Recurrence relation:

$\frac{dH_n}{dt} = 0$ can be expressed wrt both Hamiltonians

and Poisson brackets $H_1, \{,\}_1$ or $H_2, \{,\}_2$ of KdV

$$\frac{dH_n}{dt} = \{H_n, H_2\}_2 = \{H_n, H_1\}_1 = 0$$

PB 1: $$\{H_n, H_1\}_1 = \int_{-\infty}^{\infty} dx\, dy\, \frac{\delta H_n}{\delta u(x)} \left(\partial_x^3 + \frac{1}{3}(\partial_x u + u\partial_x)\right) \delta(x-y) \underbrace{\frac{\delta H_1}{\delta u(y)}}_{=u(y)}$$

$$\overset{\text{integration by parts}}{=} - \int_{-\infty}^{\infty} dx\, u(x) \underbrace{\left(\partial_x^3 + \frac{1}{3}(\partial_x u + u\partial_x)\right) \frac{\delta H}{\delta u(x)}}_{\equiv\, \partial_x \frac{\delta K[u]}{\delta u(x)}}$$

$$\overset{!}{=} 0 \iff$$

because if $K[u] = \int_{-\infty}^{\infty} dx\, k[u]$ then

$$\int_{-\infty}^{\infty} dx\, \partial_x u(x) \frac{\delta K}{\delta u(x)} = \int dx\, \partial_x u(x) \frac{\partial k}{\partial u} = \int dx \frac{d}{dx} k[u(x)] = 0$$

- Scaling behaviour:

$$\left(\partial_x^3 + \frac{\partial_x u + u\partial_x}{3}\right) \frac{\delta H_n}{\delta u(x)} \quad \text{scales as} \quad \partial_x \frac{\delta K}{\delta u(x)}$$

$$\Rightarrow [K] = [H_n] - 2 \Rightarrow K \sim H_{n+1}$$

this gives the induction step in the recurrence relations

$$\boxed{D_x^3 \frac{\delta H_n}{\delta u(x)} \equiv \left(\partial_x^3 + \frac{1}{3}(\partial_x u + u\partial_x)\right) \frac{\delta H_n}{\delta u(x)} = \partial_x \frac{\delta H_{n+1}}{\delta u(x)}}$$

- we already checked the induction start for K_0, H_1, H_2

Involution = independence of H_n

$$\{H_n, H_m\}_1 = \int_{-\infty}^{\infty} dx\,dy \frac{\delta H_n}{\delta u(x)} D_x^3 \delta(x-y) \frac{\delta H_m}{\delta u(y)}$$

$$= \int_{-\infty}^{\infty} dx \frac{\delta H_n}{\delta u(x)} D_x^3 \frac{\delta H_m}{\delta u(x)}$$, use recurrence $D_x^3 \frac{\delta H_m}{\delta u(x)} = \partial_x \frac{\delta H_{m+1}}{\delta u(x)}$

$$= \int_{-\infty}^{+\infty} dx \frac{\delta H_n}{\delta u(x)} \partial_x \frac{\delta H_{m+1}}{\delta u(x)}$$

integration by parts

$$= -\int_{-\infty}^{\infty} dx\, \partial_x \frac{\delta H_n}{\delta u(x)} \frac{\delta H_{m+1}}{\delta u(x)}$$ use recurrence

$$= -\int_{-\infty}^{\infty} dx\, D_x^3 \frac{\delta H_{n-1}}{\delta u(x)} \frac{\delta H_{m+1}}{\delta u(x)} = \{H_{n-1}, H_{m+1}\}_1$$

Using iteration starting with H_0, H_1, H_2 we can show

$$0 = \{H_n, H_m\}_1 \quad \forall n,m \in \mathbb{N}$$

Similarly one can show the same statement for $\{\ \}_2$:

$$0 = \{H_n, H_m\}_2$$ exercise

Hierarchy of the KdV equation

- with the H_n in involution we have found conserved quantities for a hierarchy of nonlinear equations:

$$\frac{dH_n}{dt} = \{H_n H_m\}_{1,2} = 0$$

define $\partial_t u \equiv \{u(x), H_n\}_1 \quad (= \{u(x), H_{n+1}\}_2$ why?$)$

we then find the following equations

$\underline{n=0}$: $\quad \partial_t u = \{u(x), H_1\}_2 = \int_{-\infty}^{\infty} dz\, \partial_x \delta(x-z) \frac{\delta H_1}{\delta u(z)}$ $\quad u(z)$

$\Leftrightarrow \quad \boxed{\partial_t u = \partial_x u}$ $\quad$ chiral wave eq. $u = u(x-t)$

$\underline{n=1}$: $\quad \partial_t u = \{u(x), H_2\}_2 = \partial_x \frac{\delta H_2}{\delta u(x)}$

$\Leftrightarrow \quad \boxed{\partial_t u = u\partial_x u + \partial_x^3 u}$ $\quad$ KdV (this is how we choose $H_2, \{,\}_2$)

$\underline{n=2}$: $\quad \partial_t u = \{u(x), H_3\}_2$

leads to $\quad \underline{\partial_t u = \partial_x^5 u + \frac{4}{3} u \partial_x^3 u + \frac{10}{3} \partial_x u \partial_x^2 u + \frac{5}{6} u^2 \partial_x u}$

(exercise)

- this infinite tower of integrable non linear PDE's is called <u>KdV Hierarchy</u>.