The Inverse Scattering Method

Idea: we interpret $u(x,t)$ as the potential of a time indep Schrödinger eq. Given the initial cond. $u(x, t=t_0)$ and that u satisfies KdV $\forall$ times t

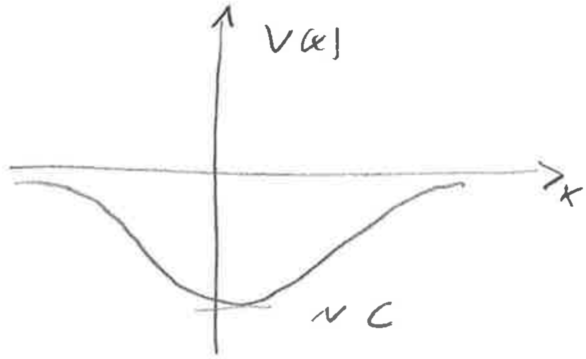
$$\partial_t u(x,t) = u\,\partial_x u(x,t) + \partial_x^3 u(x,t)$$

we can deduce $u(x,t_0) \Rightarrow u(x,t)$ the sol. of KdV $\forall t$

recall $$u(x,t) = \frac{3c}{\cosh^2\left[\frac{\sqrt{c}}{2}(x+ct)\right]}$$

Schrödinger eq. $H\Psi = E\Psi$:

$$\boxed{\begin{array}{l} -\dfrac{\partial^2}{\partial x^2}\Psi + V(x)\Psi = \lambda\Psi \\ \text{with } V(x) = -\dfrac{1}{6}u(x,0) \end{array}}$$



$\Psi(x;t)$ is the wave function, depending on parameter t
λ the energy eigenvalue

* Note: t is not Schrödinger time $\tau \neq t$

time dep. Schrödi $\quad i\hbar \dfrac{\partial}{\partial \tau}\Psi = H\Psi$

* the choice of $c>0$ varies the depth of potential V
$\Rightarrow$ varies the number n of bound states with energy $\lambda < 0$

* We will first determine the t-dependence of $\Psi(x;t)$ and then reconstruct the full $u(x,t)$ from $\Psi(x;t)$ using init. cond $u(x,t_0)$

remark: • We shall study this inverse scattering method for the simplest 1 soliton example.
Here we could also have simply used Galilei invariance

$$u(x,0) \sim \frac{1}{\cosh^2(x)}, \quad \text{solution of kdV } f(x+ct)$$

$$\Rightarrow u(x,t) \sim \frac{1}{\cosh^2(x+ct)}$$

• BUT: the inv. scatt. method also works for more complicated initial conditions yielding multi-soliton solutions

Q: Does the energy eigenvalue also depend on t, $\lambda = \lambda(t)$?

* λ is x-indep. being a constant (energy) for the 1D Schrödi

• assume $\psi(x;t) \neq 0$: divide Schrödi by ψ.

$$\Rightarrow \quad \underline{u(x;t) = -6\left(\lambda + \frac{\partial_x^2 \psi}{\psi}\right)}$$

insert into kdV.

$$\partial_t u(x,t) = -6\left(\frac{\partial \lambda}{\partial t} + \frac{\psi \partial_t \partial_x^2 \psi - \partial_t \psi \partial_x^2 \psi}{\psi^2}\right)$$

$$\partial_x u(x,t) = (-6)\left(\frac{\psi \partial_x^3 \psi - \partial_x \psi \partial_x^2 \psi}{\psi^2}\right) = \ldots = \frac{-6}{\psi^2} \partial_x\left[\psi^2 \partial_x\left(\frac{\partial_x \psi}{\psi}\right)\right]$$

$$\partial_x^2 u = \ldots, \quad \partial_x^3 u = \ldots$$

insert into kdV & multiply by ψ^2 $\Rightarrow$ (exercise!)

$$\underline{0 = \psi^2 \partial_t \lambda - \partial_x\left[\psi^2 \partial_x\left[\frac{1}{\psi}\left(\partial_x^3 \psi - \partial_t \psi + \left(\frac{u}{2} - 3\lambda\right)\partial_x \psi\right)\right]\right]}$$

• integrating $\int_{-\infty}^{\infty} dx$, using that $\int_{-\infty}^{\infty} dx\, \psi(x)^2$ is finite ("norm of wave-funct.")

λ is x-indep. and Ψ and its $\partial_x^n \Psi$ vanish at $\pm\infty$ (b.c.)

we obtain $\boxed{\partial_t \lambda = 0}$ the energy λ is x- and t-indep.!

- this implies $\Psi^2 \partial_x\left[\frac{1}{4}\left(\partial_x^3\Psi - \partial_t\Psi + \left(\frac{u}{2} - 3\lambda\right)\partial_x\Psi\right)\right] = \text{const}$ $\forall x,t$

using the b.c. for $\left\{\begin{matrix} u(x,t) \\ \Psi(x\to\pm\infty)\end{matrix}\right\}$ deduce the const $\equiv 0$ $\boxed{\partial_x\left[\text{——— ” ———} \right] = 0}$

- ∂_x (Schrödi): $-\partial_x^3\Psi = +\lambda\partial_x\Psi + \frac{1}{6}\left(\partial_x u \Psi + u \partial_x \Psi\right)$

$$\Rightarrow \boxed{\partial_x\left[\frac{1}{4}\left(-\partial_t\Psi - 4\lambda\partial_x\Psi - \frac{1}{6}\partial_x u \Psi + \frac{1}{3} u \partial_x \Psi\right)\right] = 0} \quad (*)$$

$\Rightarrow$ $[\ldots] =$ const. which we'll determine using the asymptotic

The t-dependence of the wavefunction $\Psi(x,t)$

We shall distinguish bound states ($\lambda<0$) and scattering states ($\lambda>0$)

i) bound states $\quad \lambda \equiv -\kappa^2 < 0 \quad , \kappa \in \mathbb{R}$

We have as an input: $u(x,t), \partial_x u(x,t) \xrightarrow[x\to\infty]{} 0$

$\Rightarrow$ Schrödi at $x \gg 1$: $-\partial_x^2 \Psi = -\kappa^2 \Psi$ with

solution $\quad \underline{\Psi(x,t) \sim e^{-\kappa x}}$ for $x\to\infty$

consider $(*)$ for $x\to+\infty$: translate b.c. for $u(x,t)$ to b.c. for Ψ:

claim $\frac{\Psi_t}{\Psi} \to 0$ for $x\to+\infty$ (self consistently)

$$\text{const.} = \frac{1}{\psi}\Big(\psi_t + \underset{\hookrightarrow 0}{\tfrac{1}{6}u_x\psi} + 4\lambda\psi_x - \underset{\hookrightarrow 0}{\tfrac{1}{3}u\psi_x}\Big).$$

for $x \to \infty$ $\quad \hookrightarrow 0 \quad \hookrightarrow 0 \quad \hookrightarrow 0$

$$\Leftrightarrow \quad \psi\,\text{const.} = 4\lambda(-\kappa)\psi \Leftrightarrow \text{const} = 4\kappa^3$$

$$\Rightarrow \quad \psi_t + \frac{1}{6}u_x\psi + 4\lambda\psi_x - \frac{1}{3}u\psi_x - 4\kappa^3\psi = 0$$

$$\Big|\cdot\int_{-\infty}^{\infty}dx\,\psi \qquad \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}dx\,\psi^2 + \int_{-\infty}^{\infty}dx\Big(\frac{1}{6}u_x\psi^2 + 2\lambda\partial_x\psi^2 - \frac{1}{6}u\partial_x\psi^2\Big) - 4\kappa^3\int_{-\infty}^{\infty}dx\,\psi^2 = 0$$

tot. derivative

and use p 35: $\qquad$ int. by parts

$$\frac{1}{6}u_x\psi^2 = -\partial_x\Big[\psi^2\partial_x\Big(\frac{\partial_x\psi}{\psi}\Big)\Big] \quad \text{is a total derivative}$$

$$\Rightarrow \quad \frac{d}{dt}\hat{c}(t) = 8\kappa^3\hat{c}(t) \Leftrightarrow \hat{c}(t) = \hat{c}(0)e^{8\kappa^3 t} \quad \text{with}$$

$$\text{with} \quad \hat{c}(t) = \int_{-\infty}^{\infty}dx\,\psi^2(x,t) = \int_{-\infty}^{\infty}dx\,\psi^2(x,0)\,e^{8\kappa^3 t} = \int_{-\infty}^{\infty}dx\Big(\psi(x,0)e^{4\kappa^3 t}\Big)^2$$

(consistent with $\frac{\psi'}{\psi} \to 0$ for $x \to +\infty$)

ii) scattering states $\quad \lambda = k^2 \geq 0 \quad , k \in \mathbb{R}$

$$\text{plane wave} \quad \psi(k,t) \to \begin{cases} e^{ikx} + R(k,t)e^{-ikx} & x \to -\infty \\ T(k,t)e^{ikx} & x \to +\infty \end{cases}$$

with R reflection coefficient, T transmission coeff.

unitarity of scattering $\quad |R(k,t)|^2 + |T(k,t)|^2 = 1$

asymptotic behaviour from KdV: $u \to 0 \quad x \to \pm\infty$

$$(*) \quad \frac{1}{\psi}\Big(-\partial_t\psi - 4\lambda\partial_x\psi - \underbrace{\tfrac{1}{6}u_x\psi}_{\to 0} + \underbrace{\tfrac{1}{3}u\partial_x\psi}_{\to 0}\Big) = \text{const}$$

asymptot at $x \to -\infty$: reflection coeff

$$\frac{-\partial_t R\, e^{-ikx} - 4\lambda\left(ike^{ikx} - ikRe^{-ikx}\right)}{e^{ikx} + Re^{-ikx}} = \text{const} \quad \text{wrt } x$$

numerator = denominator:

$$\partial_t R - 4\lambda ikR = 4\lambda ikR \qquad , \ \text{const} = -4i\lambda k = -4ik^3$$

$$\Leftrightarrow \quad \partial_t R = 8ik^3 R$$

$$\Rightarrow \quad R(k,t) = R(k,0)\, e^{8ik^3 t}$$

same dispersion relation as for plane wave ansatz of linearised KdV

asymptot for $x \to +\infty$: T

$$(*) \Rightarrow \frac{-\partial_t T e^{ikx} - 4\lambda ik\, Te^{ikx}}{Te^{ikx}} = -4ik^3$$

$$\Leftrightarrow \quad \partial_t T + 4ik^3 T = 4ik^3 T \quad \Rightarrow \quad \partial_t T = 0 \ , \ T(k,t) = T(k,0)$$

Summary:

We have determined the t-dependence of the coeffs for bound states $\tilde{c}^1(t)$ and scattering states $R(t), T(t)$

- we will follow $u(x,t_0)$ init. cond $\Rightarrow R, T, c$ & their t-evolution $\Rightarrow u(x,t)$

Inverse scattering method & the Gel'fand-Levitan Eq.

- there exists a unique relation between potential $V(x)$ and wavefunction $\psi(x)$ in 1 dimensional quantum mechanics — in general it is not explicitly solvable

Gel'fand-Levitan eq.:

$$\boxed{K(x,y;t) + B(x+y;t) + \int_x^\infty dz\, K(x,z;t) B(y+z;t) = 0} \quad \text{for } y>x$$

where $B(X;t) = \underbrace{\sum_{n=1}^{P} c_n(t)\, e^{-\varkappa_n X}}_{\text{bound states}} + \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, R(k,t)\, e^{ikX}}_{\text{scattering states}}$

and $-V(X) = \frac{1}{6} u(X;t) \equiv 2\partial_X K(X,X;t)$ potential

with boundary conditions

$$\lim_{z\to\infty} \left\{ \begin{array}{c} K(x,z;t) \\ \partial_z K(x,z;t) \end{array} \right\} = 0$$

(more general than we required for $u(x,t)$)

idea of derivation:

* Fourier-trafo Schrödi $\partial_x^2 \psi + (k^2 + \frac{u}{6})\psi = 0$

$$\phi(x,\tau;t) = \int_{-\infty}^{\infty} dk\, e^{-ik\tau} \psi(x,k;t) \Rightarrow \text{diff. eq. for } \phi$$

* Ansatz $\phi_\infty(x,\tau) = \underbrace{\delta(x+\tau)}_{\text{in going}} + \underbrace{B(x-\tau)}_{\text{"out going": scattering \& bound states}}$

* show that $\phi(x) = \phi_\infty + \int_x^\infty dz\, K(x;z)\, \phi_\infty(z,\tau)$

is the full solution

$\Rightarrow$ G.-L. eq for K

<u>Derivation of G.L. eq:</u>

Schrödinger eq. for $\phi(x,\tau;t)$:

$$\boxed{\partial_x^2\phi - \partial_\tau^2\phi + \frac{1}{6}u\,\phi = 0}$$

Ansatz for asymptotic solution $\phi_\infty(x,\tau)$ for $x \to +\infty$

- use b.c. $u(x) \underset{x\to\infty}{\longrightarrow} 0 \Rightarrow \partial_x^2\phi_\infty - \partial_\tau^2\phi_\infty = 0$

$$\phi_\infty(x,\tau;t) = \int_{-\infty}^{\infty} dk\, e^{-ik(x+\tau)} + \sum_{n=1}^{P} c_n(t)\, e^{-\kappa_n(x-\tau)} + \int dk\, R(k;t)\, e^{ik(x-\tau)}$$

δ $\quad$ B

as

∂_x^2	$(-ik^2)$	$(-\kappa_n)^2$	$(ik)^2$
∂_τ^2	$(-ik^2)$	$(+\kappa_n)^2$	$(-ik)^2$

Ansatz for full solution :

$$\boxed{\phi(x,\tau;t) = \phi_\infty(x,\tau;t) + \int_x^\infty dz\, K(x,z;t)\,\phi_\infty(z,\tau;t)} \quad \text{for } z > x$$

we have

- $\partial_\tau^2\phi(x,\tau) = \partial_\tau^2\phi_\infty(x,\tau) + \int_x^\infty dz\, K(x,z;t)\, \partial_\tau^2\phi_\infty(z,\tau;t)$

- $\partial_x^2 \phi(x,\tau;t) = \partial_x^2 \phi_\infty + \partial_x \left[-K(x,x;t)\phi_\infty + \int_x^\infty dz\, \partial_x K(x,z;t)\, \phi_\infty \right]$

$$= \partial_x^2 \phi_\infty - \partial_x K(x,x;t)\,\phi_\infty - K(x,x;t)\,\partial_x \phi_\infty - \partial_x K(x,z;t)\Big|_{z=x} \phi_\infty + \int_x^\infty dz\, \partial_x^2 K(x,z;t)\,\phi_\infty$$

- Use Schrödi for ϕ and asymptot. Schrödi for ϕ_∞:

$$0 = \partial_x^2 \phi - \partial_\tau^2 \phi + \frac{u}{6}\phi = \partial_x^2\phi - \partial_\tau^2 \phi + 2\partial_x K(x,x;t)\phi$$

$$= \partial_x^2 \phi_\infty - \partial_x K(x,x;t)\,\phi_\infty - K(x,x;t)\,\partial_x\phi_\infty - \partial_x K(x,z;t)\Big|_{z=x}\phi_\infty$$

$$+ \int_x^\infty dz\, \partial_x^2 K(x,z;t)\,\phi_\infty - \partial_\tau^2\phi_\infty \underbrace{- \int_x^\infty dz\, K(x,z;t)\, \partial_\tau^2 \phi_\infty(z,\tau;t)}_{=:\, I}$$

$$+ 2\left(\partial_x K(x,x;t)\right)\left(\phi_\infty + \int_x^\infty dz\, K(x,z;t)\,\phi_\infty\right)$$

Consider $I = -\int_x^\infty dz\, K(x,z;t)\, \partial_z^2 \phi_\infty(z,\tau;t)$ using asymptot. Schrödi:

$$= + \int_x^\infty dz\, \partial_z K(x,z;t)\, \partial_z \phi_\infty(z,\tau;t) - \left[K(x,z;t)\,\partial_z \phi_\infty(z,\tau)\right]_x^\infty$$

$$= - \int_x^\infty dz\, \partial_z^2 K(x,z;t)\,\phi_\infty(z,\tau;t) + \underbrace{\left[\partial_z K\, \phi_\infty\right]_x^\infty}_{= -\left(\partial_z K(x,z;t)\right)\phi_\infty \big|_{z=x}}$$

$$+ K(x,x;t)\,\partial_x \phi_\infty$$

Cancelling all terms we get

$$0 = \int_x^\infty dz \Big[\partial_x^2 K(x,z;t) - \partial_z^2 K(x,z;t) + \underbrace{2\left(\partial_x K(x,x;t)\right)}_{\frac{u}{6}} K(x,z;t) \Big]$$

which is satisfied if

$$\boxed{\partial_x^2 K(x,z;t) - \partial_z^2 K(x,z;t) + \frac{1}{6}u(x,t)\,K(x,z;t) = 0}$$

for $z > 0$

- together with the b.c. for K, $\partial_z K$ this provides a unique solution for $K(x,z;t)$
- use properties of ϕ to get G-L eq:

$$\phi(x,\tau;t) = 0 \quad \text{for} \quad (x+\tau) < 0 \quad (\text{time } \tau \text{ before } S(x+\tau) \text{ comes in})$$

$$\Rightarrow \phi(x,\tau;t) = \phi_\infty(x,\tau;t) + \int_x^\infty dz\, K(x,z;t)\, \phi_\infty(x,\tau;t) = 0 \quad \text{for } (x+\tau)<0$$

we also have $\phi_\infty(x,\tau;t) = S(x+\tau) + B(x-\tau)$

$$\Rightarrow \underbrace{S(x+\tau)}_{=0 \text{ for } x+\tau<0} + B(x-\tau) + \int_x^\infty dz\, K(x,z;t)\left(S(z+\tau) + B(z-\tau)\right) = 0$$

0 for $x+\tau<0 \Leftrightarrow x<-\tau \Rightarrow \exists z>x$ s.th. $z=-\tau$

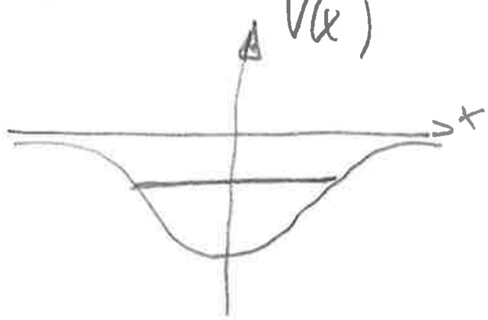
$$\Rightarrow \boxed{B(x-\tau;t) + K(x,-\tau;t) + \int_x^\infty dz\, K(x,z;t)\, B(z-\tau) = 0}$$

choosing $y=-\tau$ we obtain the Gel'fand-Levitan eq.

- <u>Application of G.-L. eq to the KdV</u>

How to get $u(x,t)$ from $u(x,0)$?

* choose the potential $V(x) = -\frac{1}{6} u(x, t=0)$ s.th there is only 1 bound state (why later)



- use that the KdV is dispersion less

  = reflection coeff $R(k,t) = 0 \quad \forall t$

  ($\Leftrightarrow$ KdV has only travelling sol. $f = f(x+ct)$ (($\Rightarrow$ shift to $t \neq 0$ trivial)))

* Note that only solutions of G.-L. for $R \equiv 0$ have been found in the literature!

inserting $p=1$ & $R \equiv 0$ we have

$$B(x+y;t) = c(t)\, e^{-\varkappa(x+y)}$$

KdV analysis of bound states gave $c^{-1}(t) = \int_{-\infty}^{\infty} dx\, \psi^2(x,t)$

and $c(t) = c(0)\, e^{-8\varkappa^2 t}$

inserting $\boxed{B(x+y;t) = c(0)\, e^{-8\varkappa^2 t} e^{-\varkappa(x+y)}}$ into

the Gel'fand - Levitan eq we can determine

$K(x,y;t)$ for any t

- $\Rightarrow$ t-dependence of $u(x,t) \cdot \frac{1}{6} = 2\partial K(x,x;t)$

we will now explicitly solve the G.L. eq to achieve this