1 soliton solution of the G.-L. equation

we have $u(x,t=0) = \dfrac{3c}{ch^2(\frac{\sqrt{c}}{2}x)} = -6V(x)$

and $-\partial_x^2\psi + V(x)\psi = \lambda\psi$, for boundstates $\lambda = -\kappa^2$

$$\Leftrightarrow \quad \boxed{\partial_x^2\psi + \frac{c}{2}\cdot\frac{1}{ch^2(\frac{\sqrt{c}}{2}x)}\psi = \kappa^2\psi}$$

<u>Step 1:</u> find $\psi(x,t=0)$ to determine $c(0)^{-1} = \int_{-\infty}^{\infty} dx\, \psi^2(x,0)^2$ in $B(x+y;t)$

<u>Ansatz:</u> $\psi(x,0) = \frac{1}{2}\dfrac{1}{ch(\kappa x)} \rightarrow \begin{cases} e^{-\kappa x} & x\to+\infty \\ e^{+\kappa x} & x\to-\infty \end{cases}$ which is a bound state (solving Schrödi asymptot.)

- $\partial_x\psi(x) = -\dfrac{\kappa}{2}\dfrac{sh(\kappa x)}{ch^2(\kappa x)}$

$ch^2 - sh^2 = 1$

- $\partial_x^2\psi(x) = -\dfrac{\kappa^2}{2}\left(\dfrac{\cancel{ch(\kappa x)}}{ch^2(\kappa x)} - \dfrac{2\,sh^2(\kappa x)}{ch^3(\kappa x)}\right) = -\dfrac{\kappa^2}{2}\left(\dfrac{2}{ch^3(\kappa x)} - \dfrac{1}{ch(\kappa x)}\right)$

inserted into Schrödi:

$$-\frac{\kappa^2}{2}\left(\frac{2}{ch^3(\kappa x)} - \frac{1}{ch(\kappa x)}\right) + \frac{c}{4}\frac{1}{ch^2(\frac{\sqrt{c}}{2}x)}\cdot\frac{1}{ch(\kappa x)} = \frac{\kappa^2}{2}\frac{1}{ch(\kappa x)}$$

$\exists$ simple solution if $\kappa = \dfrac{\sqrt{c}}{2} \Rightarrow \dfrac{c}{4} = \kappa^2$

we have $\underline{u(x,0) = \dfrac{12\kappa^2}{ch^2(\kappa x)}}$, $\underline{\psi(x,0) = \dfrac{1}{2}\dfrac{1}{ch(\kappa x)}}$

- We will show later that $V(x) = -\frac{1}{6}u(x,0)$ has only 1 bound state for $\underline{\kappa = 1}$.

normalisation $c(0)^{-1} = \int_{-\infty}^{\infty} dx\, \Psi^2(x,0) \overset{\Psi\text{even}}{=} 2 \int_0^{\infty} dx \frac{1}{4\,ch^2(x)}$

$$= \frac{1}{2} \int_0^{\infty} dx\, \partial_x\left(\frac{sh(x)}{ch(x)}\right) = \frac{1}{2}(1-0)$$

$\Rightarrow$ p. 44: $B(x+y;t) = C(0)\, e^{-8\kappa^2 t}\, e^{-\kappa(x+y)}$

$$\overset{\kappa=1}{=} 2\, e^{-8t}\, e^{-(x+y)}$$

<u>insert into G.L. eq.</u>:

$$K(x,y;t) + 2e^{-8t-x-y} + \int_x^{\infty} dz\, K(x,z;t)\, 2e^{-8t-y-z} = 0$$

Ansatz: $\underline{K(x,y;t) = \omega(x,t)\, e^{-y}}$

$$\Rightarrow \quad \omega(x,t) + 2e^{-8t-x} + \int_x^{\infty} dz\, \omega(x,t)\, e^{-z}\, 2e^{-8t-z} = 0$$

$$\Leftrightarrow \quad \omega(x,t) + 2e^{-8t-x} + \omega(x,t)\, e^{-8t} \underbrace{\left[-e^{-2z}\right]_x^{\infty}}_{=+e^{-2x}} = 0$$

$$\Leftrightarrow \quad \omega(x,t) = \frac{-2e^{-8t-x}}{1+e^{-8t-2x}}$$

$$\Rightarrow \quad \boxed{K(x,y;t) = \frac{-2e^{-8t-x-y}}{1+e^{-8t-2x}}}$$

$\Rightarrow$ t-dependence of $u(x,t)$ via

$$u(x,t) = 12\, \partial_x K(x,x;t) = 12\, \partial_x\left(\frac{(-2)e^{-8t-2x}}{1+e^{-8t-2x}}\right)$$

$$\Leftrightarrow u(x,t) = 12\,\partial_x \left(-2 + \frac{2}{1+e^{-8t-2x}} \right)$$

$$= 12\,\frac{4\,e^{-8t-2x}}{(1+e^{-8t-2x})^2} = 12\,\frac{1}{\frac{1}{4}\left(e^{4t+x}+e^{-4t-x}\right)^2}$$

$$\Leftrightarrow \quad \boxed{u(x,t) = \frac{12}{ch^2(x+4t)}}$$

t-dependent solution of KdV

remarks: 1) we have deduced the fact that the KdV is dispersionless, i.e. $R(k,0)\equiv 0$, from knowing that the solutions of KdV take the form $f = f(x+ct)$. Of course this would have enabled us immediately to do the step $u(x,0) \to u(x,t)$

$\to$ we will revisit this by looking at the dispersion of KdV, linearised KdV and time-dep. Schrödi

2) the same G.L. eq allows to determine t-dependent, interacting multi-soliton solutions

$\to$ we will investigate the number of bound states for the KdV-potential $V(x) = -\frac{1}{6}u(x)$ as a first step into that direction

(multi-soliton later as well with Bäcklund trafo)

Properties of 1-dimensional Quantum Mechanics

- we want to study 1-dim time-indep Schrödi eq

$$\boxed{H\,\phi(x) = E\,\phi(x)} \quad \text{with} \quad \boxed{H = -\tfrac{1}{2}\partial_x^2 + \tfrac{1}{2}U(x)},$$

first for some general potential $U(x)$

* when studying the harmonic oscillator it was useful to write H in a factorised form in terms of operators "$H \sim a a^\dagger = a^\dagger a + 1$"

- Q: when can a generic H be written in a factorised form?

Ansatz $\quad A(x) = \frac{1}{\sqrt{2}}(\partial_x - W(x)), \quad W(x) \in \mathbb{R}$

$$\Rightarrow \quad A^\dagger(x) = \frac{1}{\sqrt{2}}(-\partial_x - W(x)), \quad \text{as } \partial_x^\dagger = -\partial_x$$

$$(H - E\mathbb{1}) \stackrel{!}{=} A^\dagger(x)\,A(x) = \frac{-1}{2}(\partial_x + W)(\partial_x - W)$$

$$= -\frac{1}{2}\partial_x^2 - \frac{1}{2}\left(-(\partial_x W) - W\partial_x + W\partial_x - W^2\right)$$

$$\Leftrightarrow \quad \boxed{H = -\frac{1}{2}\partial_x^2 + \frac{1}{2}\left(W^2(x) + (\partial_x W(x))\right)}$$

this equals the original H if we can find a $W(x)$ s.th.

$$\boxed{U(x) - 2E = W^2(x) + (\partial_x W(x))} \qquad \text{Riccati-type trafo}$$

* Note that for $\phi(x) \neq 0$ real these operator $A, A^\dagger$ can be constructed from this eigenfunction $\phi(x)$ as

$$\boxed{\begin{aligned} A(x) &= \frac{1}{\sqrt{2}}\,\phi(x)\,\partial_x\,\phi^{-1}(x) \\ A^\dagger(x) &= -\frac{1}{\sqrt{2}}\,\phi^{-1}(x)\,\partial_x\,\phi(x) \end{aligned}}$$

Exercise: Show using this representation that $A^\dagger(x)\,A(x)$ yields the original Hamiltonian $= H - E\mathbb{1}$ (use that $\phi(x)$ is an eigenfunction)

- as it should be for annihilation operators it holds

$$\underline{A(x)\,\phi(x)} = \frac{1}{\sqrt{2}}\phi(x)\,\partial_x\,\underbrace{\phi^{-1}(x)\,\phi(x)} = \underline{0}$$

and likewise $\underline{A^\dagger(x)\,\phi^{-1}(x)} = -\frac{1}{\sqrt{2}}\phi^{-1}(x)\,\partial_x\,\hat{\phi}(x)\,\phi^{-1}(x) = \underline{0}$

$*$ Note that this leads to a formal solution for $\phi(x)$:

$$A(x)\,\phi(x) = \frac{1}{\sqrt{2}}(\partial_x - W(x))\,\phi(x) = 0$$

which is solved by $\underline{\phi(x) = \exp\left[\int^x dy\, W(y)\right]}$

Analysis of the number of bound states

- Let us consider $H_+ \equiv A^\dagger(x)A(x) = H - E\mathbb{1}$

$$\boxed{H_+ = -\frac{1}{2}\partial_x^2 + \frac{1}{2}u(x) - E}$$

$$= -\frac{1}{2}\partial_x^2 + \frac{1}{2}W^2(x) + \frac{1}{2}(\partial_x W(x))$$

Defining the operator $H_- \equiv A(x)\,A^\dagger(x)$

one can easily show

$$= -\frac{1}{2}(\partial_x - W)(\partial_x + W)$$

$$= -\frac{1}{2}\partial_x^2 + \frac{1}{2}W^2(x) - \frac{1}{2}(\partial_x W(x))$$

exercise: it also holds $H_- = -\frac{1}{2}\partial_x^2 + \frac{1}{2}\tilde{u}(x) - E$

with $\hat{u}(x) = u(x) - 2\left(\partial_x^2(\ln\phi(x))\right)$

• We will now show that H_+ and H_- are almost isospectral ($\equiv$ have the same eigenvalues) to deduce the # of bound states in KdV:

$\Psi(x)$ eigenfunction of H_+: $H_+\Psi(x) = A^\dagger(x)A(x)\Psi(x) = \epsilon\Psi(x)$

$$\Rightarrow A(x)A^\dagger(x)A(x)\Psi = \epsilon A(x)\Psi(x)$$

$$\Leftrightarrow H_-(A(x)\Psi) = \epsilon(A(x)\Psi(x))$$

so if Ψ is an eigenstate of H_+ with eigenvalue ϵ

$A\Psi$ is an " " H_- " " ϵ

unless $\boxed{A(x)\Psi(x) = 0}$

$\Rightarrow$ H_+ and H_- are almost isospectral

Application to KdV:

choose $\frac{1}{2}U_n(x) = -\frac{n(n+1)}{2}\frac{1}{ch^2(x)}$ with $n \in \mathbb{N}$

so that $H_n = -\frac{1}{2}\partial_x^2 - \frac{n(n+1)}{2}\frac{1}{ch^2(x)}$

• for this $U_n(x)$ we can find $W_n(x)$ as follows:

$W_n(x) \overset{!}{=} -n\frac{sh(x)}{ch(x)}$ $\Rightarrow$ $\partial_x W_n = -n\left(\frac{-sh^2(x)}{ch^2(x)} + \frac{ch(x)^2}{ch(x)^2}\right)$

$= -\frac{1}{ch^2(x)}$

$\Rightarrow W_n^2(x) + (\partial_x W_n(x))$

$$= n^2\frac{sh^2}{ch^2} - n\frac{1}{ch^2} \underset{\uparrow}{=} n^2 - n(n+1)\frac{1}{ch^2(x)} \overset{!}{\underset{\text{p.48}}{=}} U_n(x) - 2E_n$$

$sh^2 = ch^2 - 1$

$\Leftrightarrow$ $\boxed{E_n \equiv -\frac{n^2}{2}}$

$$\Rightarrow \quad \boxed{H_+ = H_n - \left(-\frac{n^2}{2}\right)}$$

and $H_- = -\frac{1}{2}\partial_x^2 + \frac{1}{2}\Big(\underbrace{W_n^2(x) - (\partial_x W_n(x))}\Big)$ are almost isospectral

from previous page $\quad = n^2 - n(n-1)\frac{1}{ch^2(x)}$

$$\Leftrightarrow \quad \boxed{H_- = H_{n-1} - \left(-\frac{n^2}{2}\right)}$$

so H_n and H_{n-1} are iso-spectral except for the eigenvalue with $\left(H_n + \frac{n^2}{2}\right)\Psi_n(x) = 0 \quad (\Rightarrow A(x)\Psi(x) = 0)$
so they don't share eigenvalue $\frac{n^2}{2}$

- likewise we can show that H_{n-1} and H_{n-2} are almost isospectral except for eigenvalue $\frac{(n-1)^2}{2}$ etc.
- when arriving at $n = 0$ we have the free Hamiltonian $H_{n=0} = -\frac{1}{2}\partial_x^2$ with no discrete eigenvalues

$\Rightarrow$ H_n has the following discrete energy eigenvalues

$E_k = -\frac{k^2}{2}$ with $k = 1, 2, \ldots, n$ which are call bound states

$$\Rightarrow \quad \boxed{\frac{1}{2}U_n(x) = -\frac{n(n+1)}{2}\frac{1}{ch^2(x)}} \text{ has } n \text{ bound states,}$$

in particular $\frac{1}{2}U_1(x) = -\frac{1}{ch^2(x)}$ has 1 bound state

$\Leftrightarrow U_1(x) = V(x) = -\frac{2}{ch^2(x)} = -\frac{1}{6}\cdot\frac{12}{ch^2(x)}$ compared to page 45